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Event-specific multiplicative cascade models and an application to rainfall

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Abstract. Multiplicative cascades offer parsimonious models capable of capturing the scale-invariant (multifractal scaling) behavior of some geophysical phenomena, such as rainfall, over a large range of scales. While these models achieve a remarkable degree of universality, it is still unclear how to characterize individual events within this framework. The present work offers an event description based on a few most important (amplitude-wise) branchings of the event's multiplicative cascade generator. The proposed method is based on the modulus extrema of wavelet transforms and indexes the branches (or generator weights in the multiplicative cascade model) such that their number at each branching, magnitude, and the relative scales at which they occur can be extracted and memorized. In this way, a particular event can be characterized in a multiplicative cascade framework by only a few significant weights and their respective positioning within the cascade. The application of the present model to rainfall is supported by the evidence of branching of the wavelet modulus extrema as well as by the findings [*Venugopal and Foufoula-Georgiou*, 1996; *Cârsteanu et al.*, 1997] that an important part of the signal energy of temporal rainfall events can be recovered from a few wavelet-packet components.

1. Introduction

Multiplicative cascades generate measures defined on the appropriate support (e.g., a surface and/or a time axis), showing scale invariance in the form of multifractality or, in particular cases, monofractality. The main motivation in modeling processes with multiplicative cascades is to capture the scale-invariant behavior of the process when present, and hereby obtain a parsimonious description over several scales. For overviews of multiplicative cascade models and multifractal measures, see, for example, *Gupta and Waymire* [1990], *Lovejoy and Schertzer* [1991], and *Davis et al.* [1994]. Cascades can be constructed in terms of an infinite iterative procedure. Beginning with a given "mass" uniformly distributed over the support, each subsequent step divides the support and generates a number of weights (called the "branching number" of the generator), such that the mass is redistributed to each branch by multiplication with the respective weight (Figure 1, top).

To achieve conservation in the ensemble average of the mass, the expected value of the sum of weights at each branching should be equal to unity. If the sum of the weights at every branching is exactly equal to unity, then weights are said to be complementary ("microcanonical" cascade). Note that once a process is assumed to be described by a cascade, the available measurements are treated as aggregates of the infinitely cascaded values. These aggregates (the reconstructed or "dressed" cascade, see Figure 1, bottom) are equal to the cascading values (the "bare" cascade) only in the case of complementary-weights cascades. Moreover, the measurable quantity, which is seen as the result of the cascading process over a large number of aggregation levels, could be a tracer in the cascading process and not the cascading quantity itself, possibly related to the cascading quantity by a power law at the smallest scales [*Schertzer and Lovejoy*, 1987].

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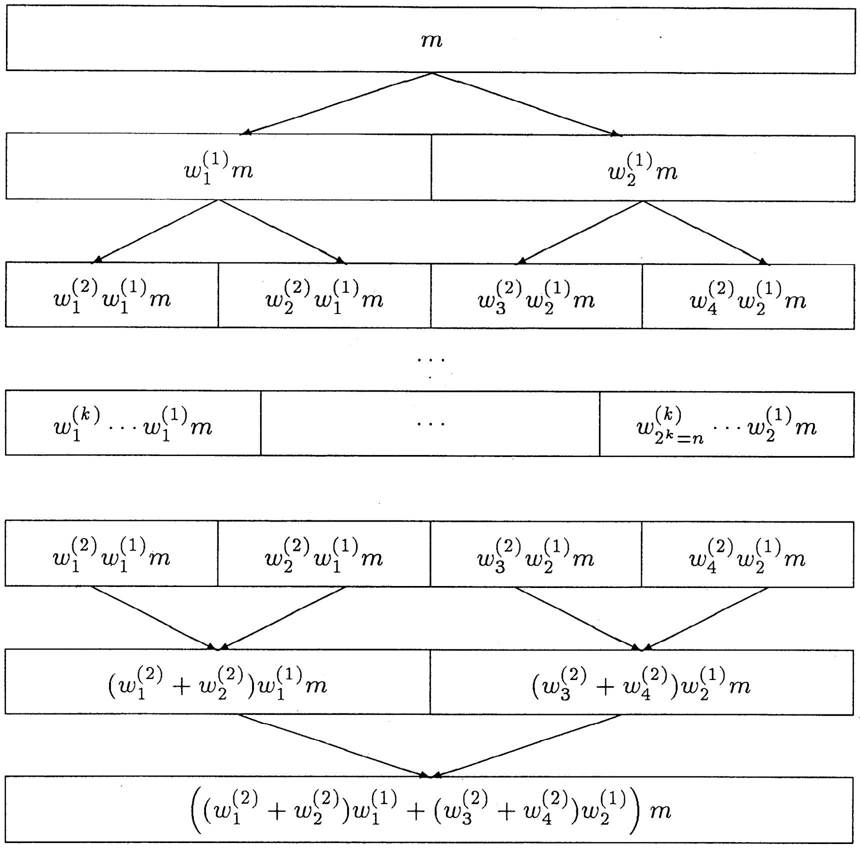


Figure 1. A "bare" cascade, i.e., the levels of generating a multiplicative cascade (top), and the last three levels of reconstructing the cascade, the simplest "dressed" cascade, from those small-scale values (bottom). Notice that in the generating mechanism the quantity m contained on the whole support is first "split" between the two halves of the support according to the respective weights w, then the quantity in each half is split between quarters, and so on. Notice that when reconstructing the cascade, the "weights" that can be defined by analogy with the generating process, i.e., the ratio between each quantity and its "parent" from which it split, will not have the same values as in the generating process, unless each pair of weights is exactly complementary, i.e., sums up to 1. Note also that the same procedure which is illustrated here on a one-dimensional (1-D) support can be applied in 2-D and 3-D.

The cascade generator properties (branching number, support division, as well as finite-dimensional joint and marginal distributions of weights) are overdetermining the cascade, in the sense that certain combinations of them lead to identical cascades, and some other combinations lead to asymptotically identical cascades (i.e., as the cascading process goes to infinity, the generated cascades tend to become identical). Thus there is a tendency to believe that certain parameters, such as the branching number and the support division, can be arbitrarily chosen when building a multiplicative cascade model, and only the weights distribution should be of concern. However, over a finite range of scales generally available from a process, it is shown here that assuming different branching numbers may lead to different properties of the reconstructed cascade. Most importantly, depending on the properties of the weights distribution, it can even obscure the presence of scale invariance in the cascade. The influence of the branching number is shown to be felt if the scale-invariant properties of the cascade occur at a discrete set of scales, as opposed to occurring over a dense range of scales, in which case there is no preferential branching number.

This paper examines modeling choices of the cascade generator properties, using as an example temporal rainfall series. It proposes a method, based on the modulus extrema of continuous wavelet transforms, which identifies whether for a given series the choice of branching number is critical and, if so, determines it uniquely. It is further shown how in the context of a cascade model (using the respective branching number), a discrete equivalent of the proposed wavelet-based method can be used to select a few most significant weights and their position within the cascade. The reconstruction process using only these few weights and the statistics of generator weights is shown to be remarkably successful in reproducing temporal rainfall series in our case.

2. Modeling Choices for Cascade Generator Properties

Customarily, at each step of the cascade, a branching number is assumed, as well as an equal division of the support by the branching number. The most widely used branching number is thus $b = 2^d$, where $\mathbf{R}^d$ is the cascade support. Then, with the assumed branching number and support division, the weights distribution of the reconstructed ("dressed") cascade can be computed, and it can be checked whether it is scale invariant. It is important to notice though that when reconstructing a cascade in the presence of discrete scale invariance (i.e., scale invariance over a discrete set of scales only), the misspecification of the assumed branching tree, comprising branching number and support division, may hide the existing scale invariance in a process.

To illustrate this fact, a ternary multinomial cascade (i.e., a cascade with branching number equal to 3 and multinomial distribution of weights) was used to generate a series. This series was then used to reconstruct the cascade binarily. Notice here that although the true underlying branching number is 3, we assumed a branching number of 2 in the reconstruction. The marginal cumulative distribution functions (CDFs) of weights at each level of the reconstructed cascade (each level corresponding to a different scale) were then estimated and are displayed in Figure 2. It is noted from Figure 2 that the differences between the CDFs at different levels (scales) can be misconstrued into an indication of

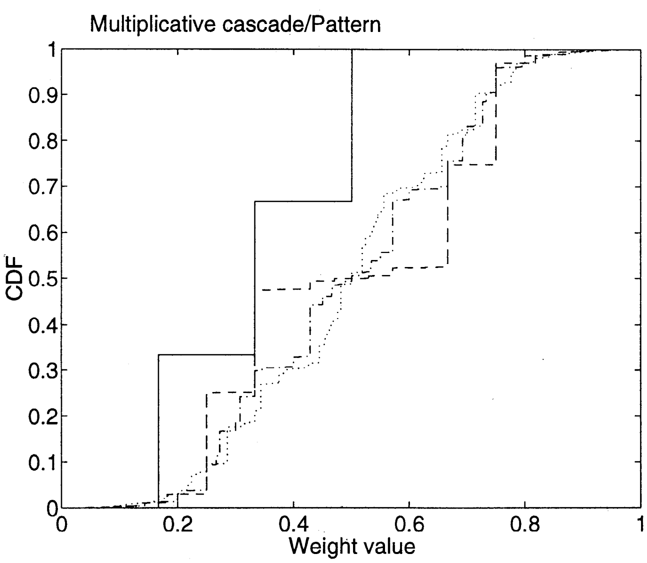


Figure 2. Marginal distributions of weights in a ternary complementary-weights multinomial cascade (solid line), as well as in the reconstruction of the same as a binary cascade, at the first (dashed line), second (dashed-dotted line), and third (dotted line) level of reconstruction. To observe are the differences between the distributions at the different levels of reconstruction, showing that the binarily reconstructed cascade is not scale invariant.

lack of scale invariance of the process, although here, clearly, the complementary-weights, ternary cascade is built in a strictly scale-invariant manner (complementary weights also imply that the generated cascade is identical to the reconstructed cascade; see, for example, *Cârsteanu and Foufoula-Georgiou* [1996] and *Olsson* [1998] for further discussion of this issue).

It should be thus observed that in the case of the presence of discrete scale invariance (a) the branching number is not trivial to determine from the data and (b) detecting the presence of scale invariance over a finite number of scales may depend on using the correct (unknown) branching number. This is the reason why a tool is needed to determine the branching number from a data set that has supposedly been generated by a cascading process.

For a given data series the marginal weights distribution is dependent, as shown above, on the chosen branching number. When the branching number is established, to every generated ("bare") cascade weights distribution there corresponds (noninjectively) a reconstructed weights distribution (identical to the former only if the generated cascade has complementary weights). Noninjectivity (the fact that several generated cascades can produce the same reconstructed cascade) could favor the idea of using a complementary-weights cascade for modeling purposes in the first place [*Olsson*, 1998], unless there are strong reasons to favor the selection of a particular noncomplementary weights generator, such as universality criteria [e.g., *Lovejoy and Schertzer*, 1991]. Identifying a reconstructed weights distribution as being some theoretical distribution is, however simple it seems, actually not an easy task. This is because all those distributions are defined on the unit interval, are symmetric, and many of them are concentrated around 1/2 (see also Figure 9, bottom). Also, the mass exponents spectra (or equivalently, the spectra of singularities) cannot be used for that purpose, because their shape often does not depend significantly on the marginal weights distributions of the reconstructed cascades [*Cârsteanu*, 1997]. Moreover, very little is known about the joint distributions of weights and their effects on the reconstructed cascades. For example, a study by *Cârsteanu and Foufoula-Georgiou* [1996] linked the autocorrelation of cascade weights to the oscillation patterns of the produced series and demonstrated that multiplicative cascades with negative autocorrelation in their weights are more appropriate for modeling temporal rainfall series than cascades with independent weights.

It should also be noticed that specification of the branching number of the generator and the weights distribution still leaves one more degree of freedom related to the way the support is being divided (e.g., whether the support is being split equally or unequally and, if unequally, how it is being split). It turns out that there is a redundancy in the cascade generator properties (branching number, support division, and distribution of weights) and that the support division is equivalent to different choices of branching number/weights distribution combinations. Exploring, however, systematic ways in which support division can be chosen to unveil scale-invariant weight distributions is still an open matter but is beyond the scope of this paper.

In section 3, the wavelet transform modulus maxima method [*Muzy et al.*, 1993; *Arnéodo et al.*, 1994] is reviewed, as it is used in section 4 to develop a methodology by which the branching number of the cascade generator can be determined directly from observations.

3. Review of the "Wavelet Transform Modulus Maxima" (WTMM) Method

3.1. Characterizing Singularities Using Wavelets

The strength of a singularity of a function or a distribution f at a point x_0 is characterized by an exponent called the Hölder exponent $h(x_0)$. The Hölder exponent $h(x_0)$ is the greatest h, so there exists a constant C and a polynomial $P_n(x)$ of order n, satisfying

$$|f(x) - P_n(x - x_0)| \leq C|x - x_0|^{h(x_0)}. \qquad (1)$$

The polynomial $P_n(x)$ above corresponds to the Taylor series expansion of f around $x = x_0$. Thus $h(x_0)$ measures how irregular f is at the point x_0. The higher the exponent $h(x_0)$ the more regular the distribution f.

Typically, to characterize the regularity of a function or a distribution f, one studies the behavior of its Fourier transform $\widehat{f}$ at ∞. This, however, characterizes the global regularity (recall that Fourier transform is a global transform and is poor at space localization). Wavelets provide a framework to characterize local regularity of a function or a distribution, as explained and illustrated below.

The continuous wavelet transform (CWT) of a function f is defined as

$$T_\psi[f](b,a) = |a|^{-\frac{1}{2}} \int f(x)\psi\left(\frac{x-b}{a}\right) dx, \quad a > 0, b \in R, \qquad (2)$$

where a is the scale parameter, and b is the translation parameter. ψ defines a family of wavelets; that is, for varying values of a, wavelets of different length scales can be constructed. (For a general reference on wavelets, see *Daubechies* [1992].)

Assume that the Hölder exponent of a distribution $f(x)$ around the point $x = x_0$ is $h(x_0) \in]n, n+1[$. Then, it can be shown (see Appendix) that with an analyzing wavelet having a number of vanishing moments n_ψ greater than n, the wavelet transform of the function satisfies the following power law:

$$T_\psi[f](x_0, a) \sim |a|^{h(x_0)}. \qquad (3)$$

This says that the local singular behavior of f around x_0 is characterized by a power-law behavior of its wavelet transform, with exponent $h(x_0)$.

3.2. Wavelet Transform Modulus Maxima

Modulus maxima of the wavelet transform $T_\psi[f]$ are maxima of the modulus of $T_\psi[f](x, a_0)$, taken across x and along a particular scale a_0, i.e., the local maxima at any point x_0 such that $|T_\psi[f](x_0, a_0)| > |T_\psi[f](x, a_0)|$ on either side of x_0. Maxima lines are curves connecting the modulus maxima.

Given a wavelet ψ, with a maximum at zero, we define decaying of the wavelet as a property that as the scale $a \to 0$, $\psi \to \delta$ (Dirac) in its integral norm. Notice from (1) that if a function f has an extremum (say at x_0), then the wavelet transform (WT) of f will also have an extremum at x_0 as $a \to 0$. Hence singularities of f, which are a special case of extrema, will be characterized by extrema in the wavelet transform domain. From the continuity of ψ with a we can also conclude about the existence, in a neighborhood of x_0, of a continuous line of wavelet extrema with scale (converging to x_0 as $a \to 0$).

The capability of the wavelet transform, through its maxima, to pick out singularities of different orders is illustrated for the function $f(x) = C + |x - x_0|^{1.5}$ in Figure 3. It is clear from the functional form in the example that there exists a derivative singularity at $x = x_0$ such that $h(x_0) = 1.5$. The function, its wavelet transform with a Mexican hat wavelet (number of vanishing moments is equal to 2), and WT maxima line(s) are shown in Figure 3, where the power-law behavior of $|T_\psi[f]|$ around x_0 is depicted by a straight line in log-log coordinates. The estimated slope is $\widehat{h}(x_0) \approx 1.5$. Notice that this was expected from equation (3), since $n_\psi \geq h(x_0)$. If a wavelet with $n_\psi < h(x_0)$ were selected, then the slope would be equal to the least order larger than n_ψ for which the Taylor series of f around x_0 has a nonzero coefficient.

4. Use of the WTMM Method to Detect the Branching Structure in a Multiplicative Cascade Model

As we have seen in the previous section, the WTMM enable one to study isolated singularities. If, in addition to the maxima lines, the minima lines are also plotted, one obtains a path-connected plot. This feature is important (although the asymptotic small-scale position of minima lines, unlike that of the maxima lines, does not have any special meaning), since together the two types of lines create the bifurcations that allow the identification of the "natural" branching of the cascade. This simple observation is the basis of the proposed methodology which uses the WTMM method to detect the branching structure of a multiplicative cascade, as contemplated by *Foufoula-Georgiou* [1997].

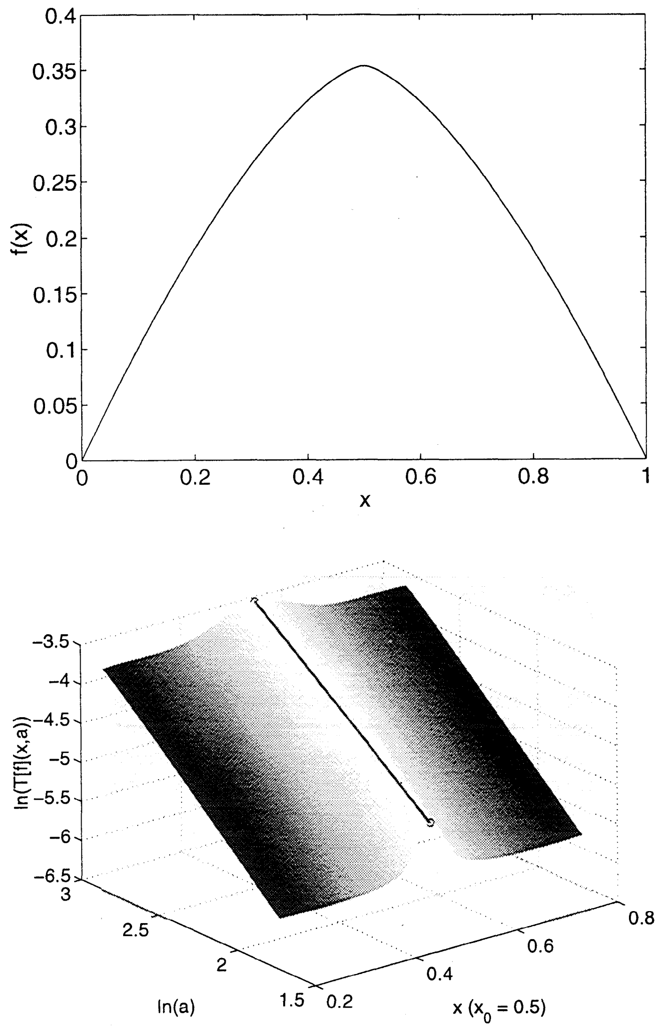


Figure 3. (top) Function $f(x) = C + |x - x_0|^{1.5}$; (bottom) the continuous wavelet transform (CWT) of the function with a Mexican hat wavelet. The modulus maxima line is also shown and its slope (= 1.5) is the value of the Hölder exponent of the function in x_0.

The sequence of maxima/minima lines along each scale, across the support, tells of the fact that the bifurcations of the extrema lines will be of either the saddle type or the branching type (see illustration in Figures 4–7). We turn our attention to fractal, specifically, self-similar measures where singularities are not necessarily isolated. A classic example is the triadic Cantor measure ($p_1 = 1/2, p_2 = 0, p_3 = 1/2$ over equal supports, in the multiplicative cascade terminology). Figure 4 shows the Cantor set and the corresponding wavelet transform modulus maxima (black) and minima (white) lines. It is evident right away that the branching structure of the cascade is captured well by the modulus extrema lines. The ratio of scales, where branching occurs, turns out to be 3. The fact that this ratio remains constant over scales indicates that we are dealing with a scale-invariant measure. In addition, the value 3, in the context of geometric self-similarity (i.e., different regions of the plot are rescaled versions of each other), signifies that we are looking at a ternary cascade.

Figure 5 shows a binary deterministic cascade ($p_1 = 1/3, p_2 = 2/3$) and the corresponding wavelet transform modulus maxima and minima (called here WT modulus extrema (WTME)) lines. Again, notice the regularity in branching and the self-similarity. The ratio of scales where branching occurs this time turns out to be 2. This correctly indicates that the underlying process could be a binary cascade.

Once the cascade is random (Figure 6) or if we deal with real-life data (rainfall time series in Figure 7), the scale invariance and self-similarity of the WTME plots become statistical in nature and are not easily discovered visually. Unlike the case of regular monofractal cascades, where the branching points are equally log-spaced along the scale axis because of their exact scale invariance and self-similarity, in the case of multifractal cascades, having an entire spectrum of singularities, branchings have an irregular character (more so, different wavelets will locally "prefer" one or another of the aforementioned singularities). We therefore need a statistical tool to infer and quantify scale invariance of the WT modulus extrema. Ideally, the tool should also indicate whether there is any indication of discrete scale invariance, such that the appropriate branching number can be chosen in that case.

Such a statistical descriptor has been found to be the ratio of the number of modulus extrema to the inverse scale (or inverse support length, if applicable) of the analyzing function. We call this descriptor "fraction of WTME" and we denote it by f_{WTME}, i.e.,

$$f_{\mathrm{WTME}} = (\text{number of WTME}) \times (\text{scale of analyzing function}). \quad (4)$$

Notice that for a rectangular-window analyzing function (Haar scaling function), the above descriptor turns out to be exactly the second-order oscillation coefficient $C_{\updownarrow}$ defined by *Cârsteanu and Foufoula-Georgiou* [1996], whose scaling behavior (for both multiplicative cascades and rainfall) has been already studied. Figure 8 shows the scale-invariant behavior of this descriptor for two multiplicative cascades and for two rainfall time series. Moreover, it shows how the descriptor picks up discrete scale invariance in a multiplicative cascade where the weights distribution is not infinitely divisible (the strong, equally spaced spikes), as opposed to the case of continuous scale invariance. The analyzed rainfall time series do not exhibit signs of discrete scale invariance.

Because no evidence of discreteness in the scale invariance of temporal rainfall has been found, we are comfortable with choosing a branching number of 2 for modeling purposes (for reasons of simplicity). Figure 9 shows weights distributions from the binary reconstruction of the Iowa City temporal rainfall data of November 1, 1990 (A). It is observed that at the first three reconstruction levels the weights distributions turn out

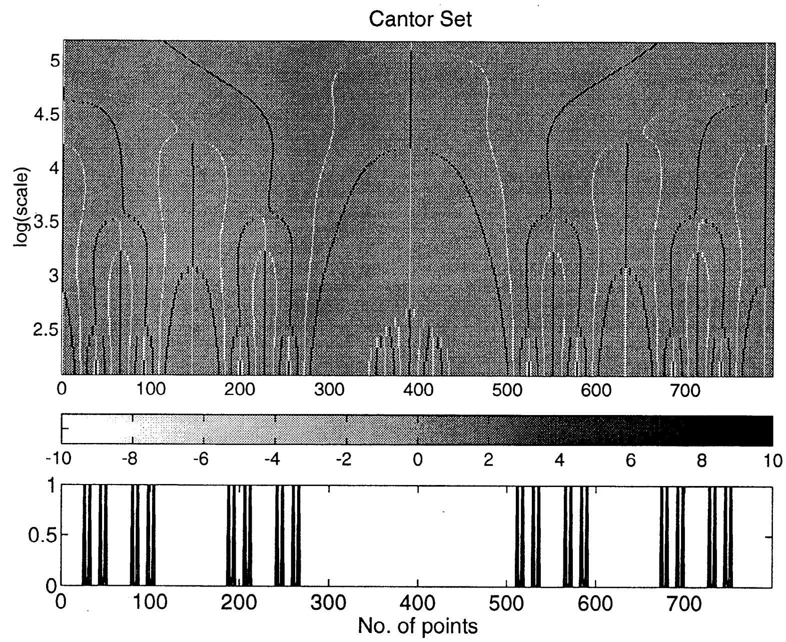


Figure 4. Analysis of the classical two-thirds Cantor set (a multiplicative cascade with weights $p_1 = 1/2$, $p_2 = 0$, $p_3 = 1/2$) with a Mexican hat wavelet. Geometric self-similarity (different regions in the plot are rescaled versions of each other) and scale invariance (ratio of scales where branching occurs remains constant over scales and equal to 3) are visible. Bifurcations are of the branching type.

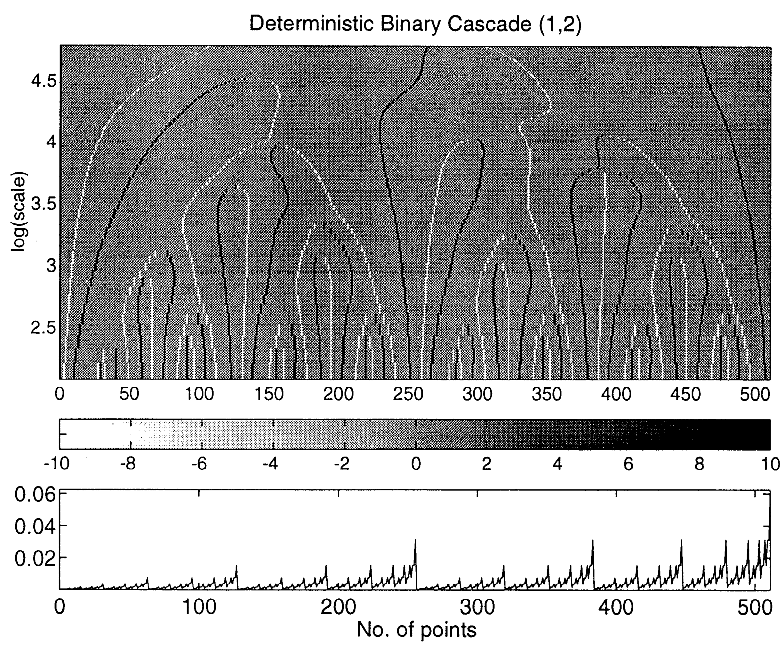


Figure 5. Analysis of a deterministic binary cascade with weights $p_1 = 1/3$, $p_2 = 2/3$, with a Mexican hat wavelet. Geometric self-similarity and scale invariance are visible. Bifurcations are of the saddle type.

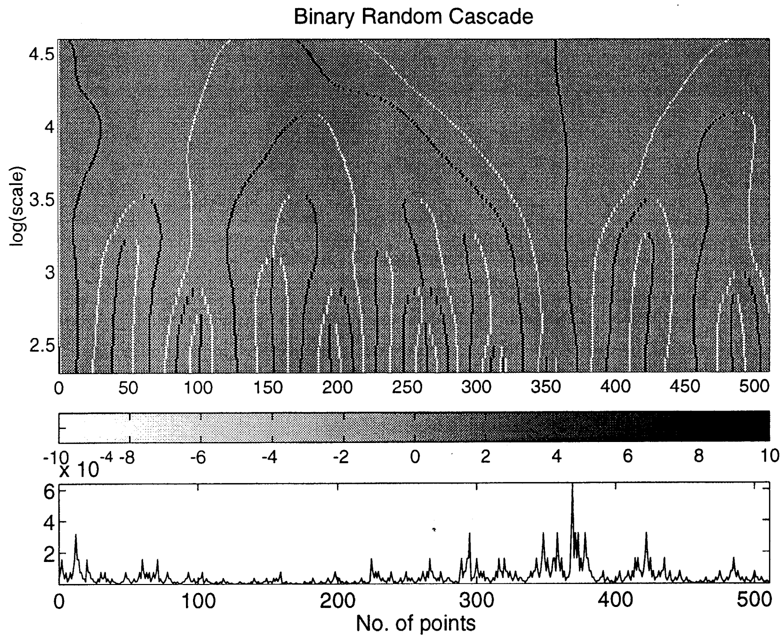


Figure 6. Analysis of a random binary, binomial cascade with weights $p_1 = 1/3$, $p_2 = 2/3$, with a Mexican hat wavelet. For a random cascade, self-similarity and scale invariance (discrete or continuous) are not obvious to the eye anymore. Both types of bifurcations appear.

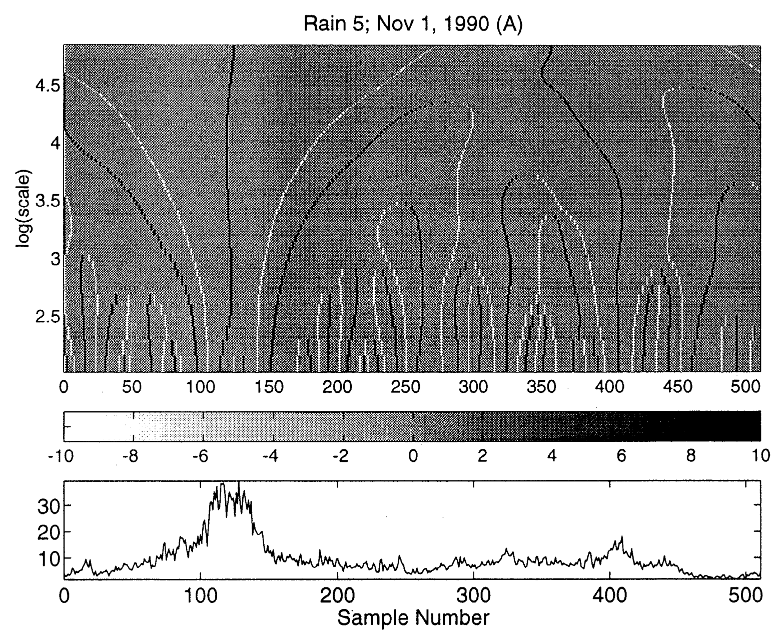


Figure 7. Modulus maxima (black) and minima (white) of the first ≈ 85 min of the April 12, 1991, Iowa City rainfall time series (bottom plot), as analyzed with a Mexican hat wavelet.

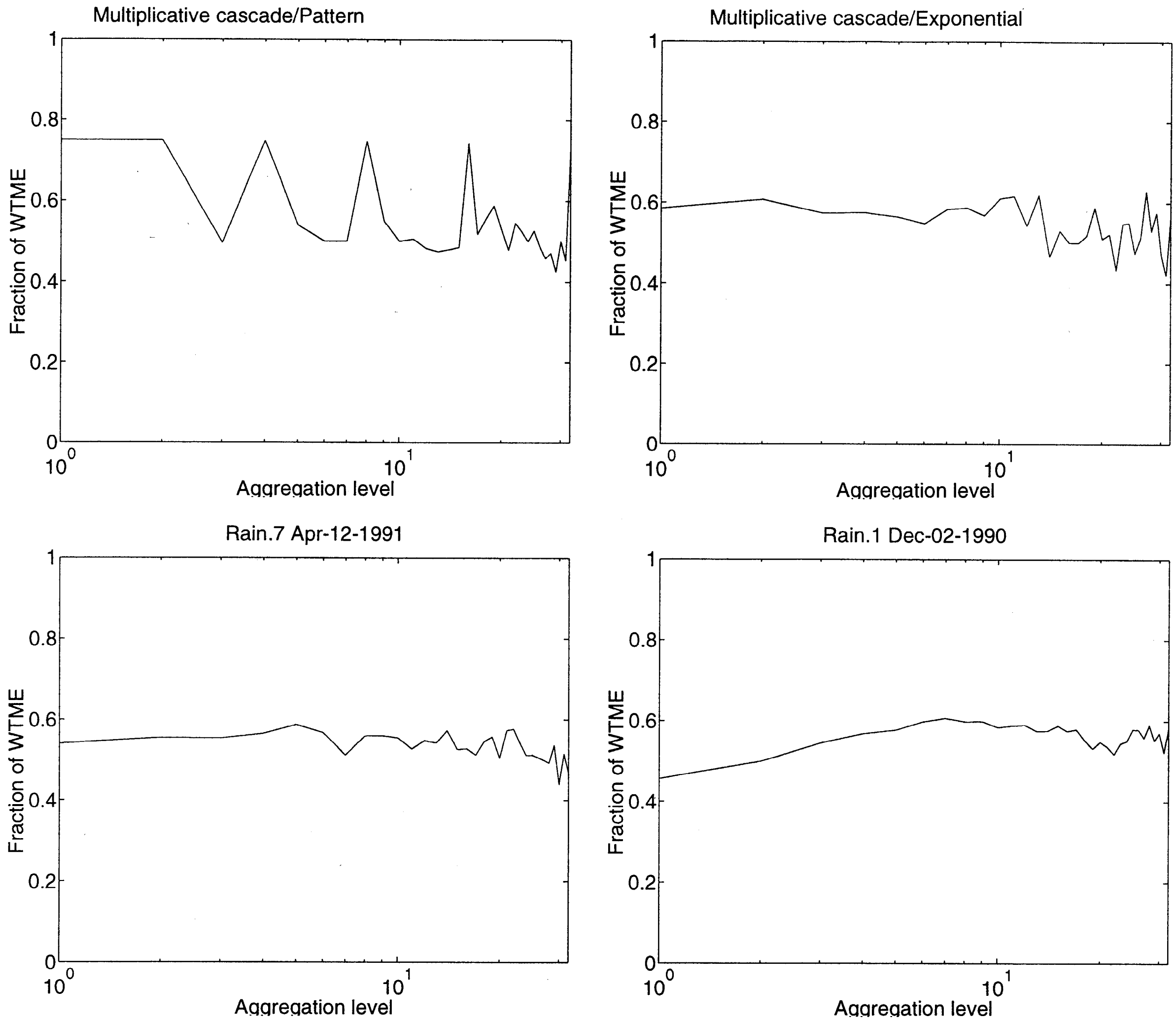


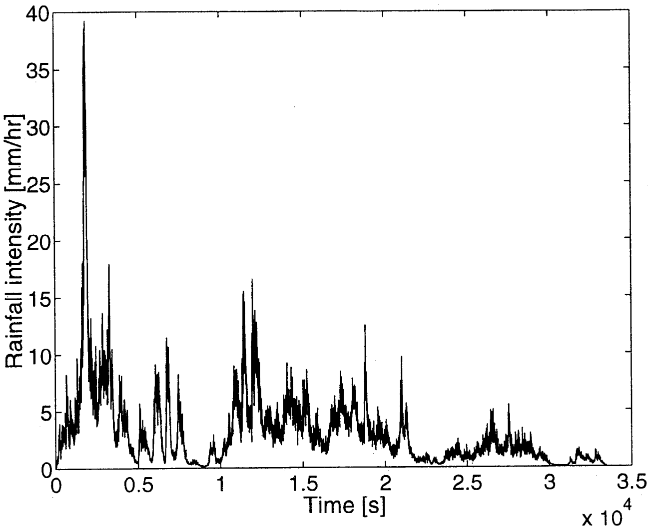
Figure 8. Wavelet transform modulus extrema (WTME) fraction, i.e., the ratio of the number of extrema to the inverse scale of the analyzing function (here the scale of the analyzing function is normalized by the signal length), for a discrete-scale (top left) and a continuous-scale multiplicative cascade (top right), and for two rainfall time series (bottom).

to be nearly identical, validating the stated choice of branching type as one of the correct choices.

5. Constructing a "Cascade Basis" From Data for Parsimonious Representation

A multiplicative cascade model fitted to a data set can be used for simulation purposes, i.e., for generation of synthetic sequences that resemble the original data in terms of its scaling properties, such as its spectrum of singularities. Possibly, more properties of the data are desired to be preserved, for example, certain regularities in the transform extrema (such as those in the deterministic cascades), or the ability is desired, to link the cascade to the dynamics of the process. This would allow one to particularize a cascade to a specific event of interest and also provide insight about the underlying generating mechanism of this process (possibly by means of its WTME skeleton). At the very least, one thing that can be done is to find the most "important" (from an amplitude point of view) branches of the cascade and hereby generate a parsimonious description of the cascading process. A method to do just that is proposed below.

To analyze a process and identify its underlying cascading mechanism, the use of a continuous wavelet (which has redundancy) is appropriate. However, for modeling purposes, where a few "model parameters" or "coefficients" are desired to be able to reproduce the statistical properties of the process, a discrete (preferably nonredundant) wavelet transform must be used. Similarly to the case of the continuous wavelet transform, for the discrete wavelet transform of a signal we can define local extrema of the transform along the support at each scale. Path connectivity in the time-



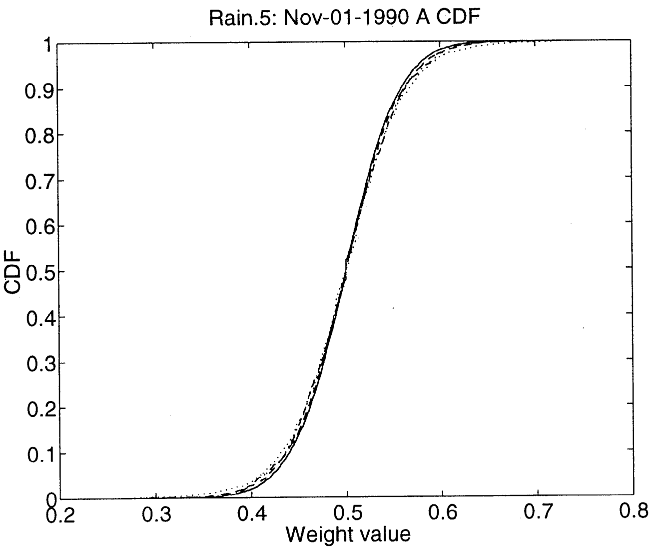


Figure 9. Weights distributions in a binary cascade reconstructed from the Iowa City temporal rainfall data of November 1, 1990(A), at the first three reconstruction levels. The distributions are almost identical at all three levels (scales) of reconstruction.

frequency plane is, of course, not a meaningful property for the discrete wavelet transform, and consequently, neither is the relation to singularities. Also, it should be noted that the branching structure is in this case predetermined (e.g., dyadic, if the "multiresolution framework" of *Mallat* [1989] is chosen). The branching number choice should be therefore inferred from the continuous WTME, as explained earlier. Notice that using a discrete cascade reconstruction for a process exhibiting scale invariance over a continuous range of scales is no contradiction, if in constructing the cascade, one uses a log-infinitely divisible distribution of generator weights.

The usefulness of the proposed procedure (i.e., identifying the most "important" branches of the cascade for a parsimonious description) lies in the identity between the time-frequency positions of the discrete wavelet coefficients and those of the weights of a multiplicative cascade. If the latter is being used to model the process, then the former can be used to identify the weights that carry the most information. Notice also that certain wavelet packet bases [see *Wickerhauser*, 1991] can accommodate a similar procedure. By sorting the most important (amplitude-wise) modulus maxima of the convolution of the process with the analyzing wavelet, one can utilize a few of the cascade weights, for example the largest ones in amplitude, for the purpose of achieving a parsimonious representation of the process.

In the case of a dyadic cascade description, retaining the most important weights amounts to using the largest dyadically reconstructed weights, with their value and position, as well as the values and positions of the weights located on the same branch, all the way down to the largest scale (see Figure 1, bottom). The process reconstruction is performed by a dyadic cascade using randomly generated complementary weights, with the exception of the retained weights, which are used with their exact values, in their respective positions.

The above procedure was applied to the rainfall series with very encouraging results. Figure 10 shows a comparison between the Iowa City rainfall of April 12, 1991, and its reconstruction from the 1, 8, and 16 largest weights. The reconstruction of rainfall events by this method shows R^2 coefficients ("explained variability") of 0.58 to 0.85 for seven Iowa City rainfall events reconstructed from the eight largest weights. These values are quite high (considering that we deal with series of 2000 to 10000 values, which were reconstructed in their entirety) and are competitive with other nonredundant representations, such as those using wavelet packet bases [e.g., *Venugopal and Foufoula-Georgiou*, 1996; *Arnéodo et al.*, 1998]. It is interesting to note that it was found by simulation that the advantage of the event-specific multiplicative cascade representations presented herein increases as the degree of multifractality in the modeled process increases. This is because the farther the process is from a nonfractal or a monofractal, the more do other nonredundant representations fail to reconstruct the process. This result is theoretically expected but difficult to quantify formally.

Selecting the few "important" weights to be retained in the reconstruction, as described in the above procedure, also drastically improves the capacity of a multiplicative cascade model to reproduce the autocorrelation function (ACF) of the process. Figure 11 shows how much closer the ACFs, obtained with the presented reconstruction method, are to the ACF of the process than the ACF of a multiplicative cascade with independent, identically distributed (i.i.d.) pairs of weights.

It should be noted that the proposed reconstruction procedure, being microcanonical in nature, preserves the scaling exponent spectra (and singularity spectra, respectively) of the original series. These scaling exponent spectra and also the weights statistics of temporal rainfall seem to be remarkably stable during each event,

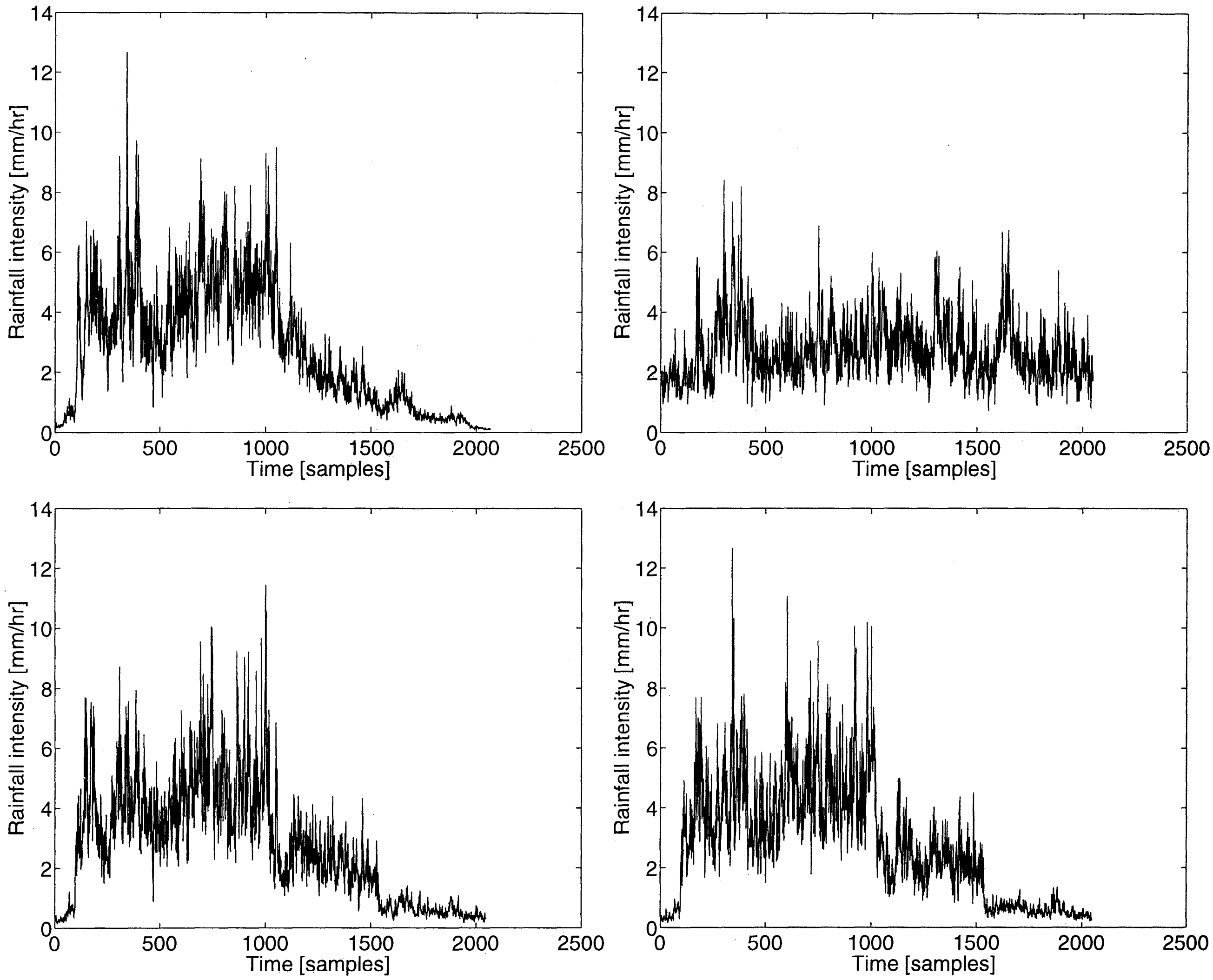


Figure 10. The April 12, 1991, Iowa City rainfall time series (top left) and its reconstruction from 1 (top right), 8 (bottom left), and 16 (bottom right) cascade weights, including the large-scale average. Notice the improved similarity of the reconstructed series to the original series as the number of selected cascade weights increases to only a few. An explained variability R^2 of 0.75 is achieved here with only 16 cascade weights.

to the point that they can be predicted from the incipient period of a rainfall event (i.e., its first few minutes), for the whole event [*Cârsteanu*, 1997].

6. Summary and Conclusions

We have shown how the wavelet transform modulus extrema (WTME) method can sensitively detect scale invariance in a process, and whether the detected scale invariance is discrete or continuous in scale. For modeling purposes the proposed method establishes the branching number (if scale invariance is discrete) in a process that exhibits a multiplicative cascading structure. Also, using a discrete version of the WTMM, we showed that a few weights of a cascade can be selected to reconstruct a temporal rainfall process, from an otherwise random nonlinear basis of weights, and the resemblance of that reconstruction to the original process is rather remarkable.

It should be noted that if one were able to relate the values and positions of the few largest weights of the cascade to prestorm physical parameters, as for example was done for the scaling exponents of rainfall fluctuations [*Perica and Foufoula-Georgiou*, 1996], the possibility would exist of using the proposed reconstruction for predictive purposes too. This, however, is beyond the scope of this paper.

Appendix: Vanishing Moments of an Analyzing Wavelet and Their Relation to Singularities of a Function

The behavior of a function f around a point x_0 can be expressed from definition (1) as

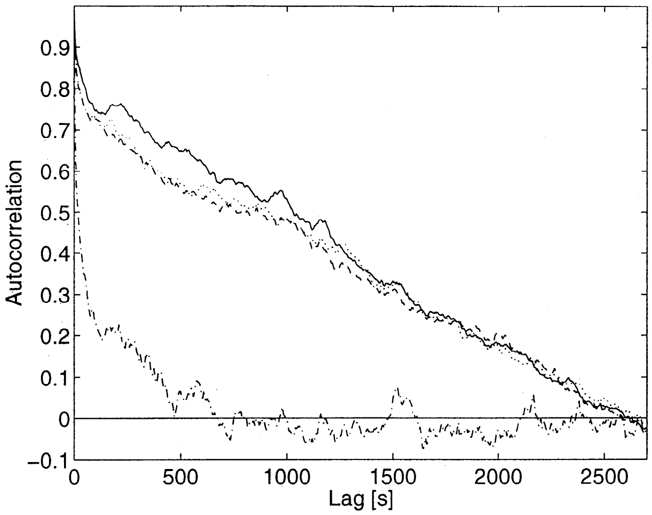


Figure 11. Decay of the autocorrelation function for the April 12, 1991, Iowa City rainfall time series (solid line) and of its reconstruction from 1 (large-scale average only, independent, identically distributed (i.i.d.) pairs of weights) (dashed-dotted line), 8 (dotted line), and 16 (dashed line) cascade weights. Notice the much better agreement with the original series as the number of selected cascade weights increases to only a few.

$$f(x) = c_0+c_1(x-x_0)+\cdots+c_n(x-x_0)^n+C|x-x_0|^{h(x_0)} \tag{A1}$$

Recall from (2) that the wavelet transform of f is the convolution of f with the family of wavelets $\psi_{a,b} = a^{-1/2}\psi\left((x-b)/a\right)$.

Thus if one assumes that the wavelet ψ has $n_\psi > n$ vanishing moments, that is,

$$\int_{-\infty}^{\infty} x^q f(x)dx = 0, \qquad 0 \le q < n_\psi, \tag{A2}$$

(note that the zeroth moment is one of the n_ψ vanishing moments), then ψ is orthogonal to polynomials up to and including order n. Hence the wavelet transform of f becomes

$$T_\psi[f](x_0,a) = \frac{1}{\sqrt{a}}C\int\psi\left(\frac{x-x_0}{a}\right)|x-x_0|^{h(x_0)}dx. \tag{A3}$$

With appropriate transformation of coordinates, it can be shown that

$$\begin{aligned} T_\psi[f](x_0,a) &= C\sqrt{a}\int\psi(x)|ax|^{h(x_0)}dx \\ &= C|a|^{h(x_0)+1/2}\int\psi(x)|x|^{h(x_0)}dx \\ &= C|a|^{h(x_0)+1/2}T_\psi[f](x_0,1). \end{aligned} \tag{A4}$$

Notice that by simply using a different normalization in the CWT, that is, $\psi_{a,b} = a^{-1}\psi\left((x-b)/a\right)$ instead of $\psi_{a,b} = a^{-1/2}\psi\left((x-b)/a\right)$ (not an L_2 norm anymore), one can reduce (A4) to

$$T_\psi[f](x_0,a) = C|a|^{h(x_0)}T_\psi[f](x_0,1). \tag{A5}$$

This is the scaling expression used in equation (3) and subsequently. It shows that the local singular behavior of f around x_0 is characterized by a power-law behavior of the wavelet transform with power exponent $h(x_0)$.

The relationship between the number of vanishing moments n_ψ of the wavelet and the Hölder exponent $h(x_0)$ can be summarized as follows: Given n_ψ, the last vanishing moment is $(n_\psi - 1)$. Thus if $h(x_0) \ge n_\psi$, the wavelet picks up the least order larger than n_ψ for which the Taylor series of f around x_0 has a nonzero coefficient. On the other hand, if $h(x_0) \le n_\psi$, the wavelet picks up $h(x_0)$, if $h(x_0)$ is fractionary, and picks up zero, if $h(x_0)$ is an integer.

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