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Assessing dependence among weights in a multiplicative cascade model of temporal rainfall

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Abstract. "Classical" multifractal analysis shows, with a good degree of confidence, that the fit of multiplicative cascades to rainfall time series is appropriate, at least from the point of view of preserving the f(α) spectrum of scaling exponents. However, basing the analysis only on the f(α) curve allows only limited discrimination between different types of cascade models; thus other descriptors with more discriminating power are needed. Also, the question of whether cascades with independent weights are appropriate for rainfall remains unanswered and needs to be addressed. In the present work we address this question and provide an assessment of the dependence structure among weights in a multiplicative cascade model of temporal rainfall. We introduce a quantity based on oscillation coefficients (describing how many of the total n-tuples of the series are obeying a certain pattern up-down-up etc.), and find that this quantity is invariant under aggregation for a multiplicative cascade model and has the ability to depict the presence and type of correlation in the weights of the cascade generator. Application of this development to high-resolution temporal rainfall series consistently suggests the need for negative correlation in weights of a binary multiplicative cascade in order to match the oscillation coefficient structure of rainfall. This is interpreted as an indication of dependence in the splitting mechanisms of intensities cascading over successive scales and might have important implications for rainfall modeling and process understanding.

1. Current Scaling Models for Rainfall

In recent years, mainly two types of scale-invariant models have proven successful in describing rainfall processes: multiscaling models of rainfall intensities based on multiplicative cascades [*Schertzer and Lovejoy*, 1987; *Gupta and Waymire*, 1991, 1993; *Tessier et al.*, 1993], and simple scaling models of rainfall wavelet fluctuations [*Kumar and Foufoula-Georgiou*, 1994; *Perica and Foufoula-Georgiou*, 1996]. A few models propose a combination of simple scaling over successive scaling regions, in long-term series of temporal rainfall intensities [*Fraedrich and Larnder*, 1993; *Olsson et al.*, 1993].

Multiplicative cascades are measures defined on the appropriate support (e.g., a surface or a time axis), showing multiple scaling defined as having a curvilinear (strictly convex) Rényi spectrum:

$$\tau(q) = \lim_{\lambda \to 0} -\frac{\ln \sum_{k=1}^{T/(\lambda \Delta t)} |r_k(\lambda \Delta t)|^q}{\ln \lambda} \qquad (1)$$

where $r(\Delta t)$ is the integral of the quantity of interest (rainfall intensity, in our case) over Δt. In the particular case of simple scaling, we have $\tau(q)$ a straight line: $\tau(q) = (1-q)\alpha$, with a unique α. Note that the transformation that relates $\tau(q)$ to the spectrum of scaling exponents f(α) is the Legendre transform (applicable under suitable assumptions on the generators [see *Holley and Waymire*, 1992]): $\alpha(q) = -d\tau(q)/dq$; $f(\alpha(q)) = q\alpha(q) + \tau(q)$ (*Frisch and Parisi* [1985], and in rainfall modeling, *Schertzer and Lovejoy* [1987]). Cascades can be described in terms of an infinite iterative construction, beginning with a given "mass" (rainfall depth in our case) uniformly distributed over the support. Each subsequent step divides the support and generates a number of weights (which is the "branching number" of the generator), such that mass is redistributed to each branch by multiplication with the respective weight. To achieve conservation in the ensemble average of the mass, the expected value of the sum of weights should be equal to unity.

Different cascade generators have been proposed for modeling rainfall. According to the probability distributions of their weights, some of the more common ones are multinomial (where the weights take a finite number of values with certain probabilities), uniform, and lognormal (see *Gupta and Waymire* [1993] for a review). The choice of the generator's probabilities in a binary, binomial cascade as a function of the large-scale average rainfall has been argued for by *Over and Gupta*

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[1994]. The lognormal cascade was first proposed by *Kolmogorov* [1962] and *Oboukhov* [1962] in the statistical theory of turbulence, and has the property that it is the limiting measure for any generator with weights whose distribution has all moments finite. A recent development in this area is the log-Poisson generator of *She and Waymire* [1995], which provides excellent agreement with the scaling observed in turbulence models. Another cascade model, used by the research groups of Lovejoy, Schertzer and Hubert, is the log-Lévy model, in which the logarithms of the weights are distributed according to a non-Gaussian stable distribution. This type of cascade produces the limiting measure for the generators with weight distributions that are attracted to Lévy-stable distributions [*Lévy*, 1937] under aggregation. A number of observations support this model versus the lognormal [see *Tessier et al.*, 1993], but (as in the case of different other weight distributions that are not suitably bounded) the lack of ergodicity of models with Lévy-exponents greater than unity [*Holley and Waymire*, 1992] also raises an estimation issue, since for real-life data it is hardly feasible to have an estimation across realizations.

Different methods can be applied to match a cascade model to rainfall time series in the sense of its (multi)scaling properties [*Lovejoy and Schertzer*, 1991]. The classical box-algorithm estimates the $\hat{\tau}(q)$ function from the data (see equation 1), and sets it equal to the theoretical $\tau(q)$ function of the model. However, multiplicative cascade generators have yet another degree of freedom: apart from the probability distribution among weights in the cascade generator, which determines the $\tau(q)$ function, the dependence structure of weights is needed to fully characterize a multiplicative cascade model. Effects of dependence in the cascade generators motivated by higher order improvements to cascade models, in the case of spatial rainfall fields has been analyzed by *Gupta and Waymire* [1995]. The mathematical consequences of a dependence among weights in multiplicative cascades is beginning to be explored [see *Waymire and Williams*, 1995], but its presence has not been investigated in data in general, and in temporal rainfall at all, in particular. It is our goal to do so, and devise the proper tools for that purpose.

2. Assessing Weight Dependence in a Multiplicative Cascade

2.1. Direct Estimation of Autocorrelation in Weights: The Binary Cascade With Complementary Weights

Throughout this work, we make use of binary (branching number 2) multiplicative cascades, which are the most parsimonious cascade models, yet are able to give rise to multifractality. Among binary cascades, those with complementary weights ($w_1 + w_2 = 1$) achieve exact (as opposed to statistical) mass conservation from one level of the cascade to the next (independently of the actual distribution of weights) and, consequently, allow the "exact" reconstruction of the underlying cascade generator directly from a data set. Therefore we can also estimate directly the autocorrelation in the weights of the generator, as well as assess the weights' distribution. Before proceeding to this end with temporal rainfall data, let us examine the behavior of the autocorrelation coefficients of lag 1 and 2 in the weights of a binary multiplicative cascade with complementary weights. A first observation is the fact that $\rho_w(1) = -(\rho_w(2) + 1)/2$ (see Appendix A for proof). Notice that $\rho_w(1)$ is always negative, an expected fact, owing to the complementarity in every second pair of consecutive weights. At the same time, when we refer to independence of weights (as far as autocorrelation is concerned), we mean the independence of adjacent pairs of weights, i.e., quantitatively $\rho_w(2) = 0$. Similarly, $\rho_w(2) \neq 0$ is referred to as dependence of weights. Notice that for $\rho_w(2) = 0$ we have $\rho_w(1) = -1/2$.

The high-resolution temporal rainfall data series used in this analysis have been collected by an optical rain gauge at the Hydro-Meteorology Lab of the Iowa Institute of Hydraulic Research in Iowa City, Iowa (see *Georgakakos et al.* [1994] for more information about the data collection). Sampling times were 5 s or 10 s for each of the seven events analyzed. The events occurred in 1990 – 1991, during different seasons (May 1990, October – December 1990, April 1991), and yet, as will be shown, they have a consistent, common behavior of the dependence in weights. A detailed analysis of these events, including distributions and power spectra, can be found in the work by *Georgakakos et al.* [1994]. As far as multifractal analysis is concerned, the log-fits of equation (1) show in all cases high correlation coefficients (see bottom plot in Figure 1 for one such case), leading to the conclusion that multiplicative cascades would make good scaling models for those rainfalls. This conclusion is also reinforced by the non-degenerate f(α) curves, of which an example is shown in Figure 1 (middle plot).

Weights have been reconstructed from the temporal rainfall series $(r_1, r_2, ...)$ as $w_i = r_i/(r_i + r_{i+1})$. Notice here that in reconstructing a cascade, we have no criterion as to which values to pair together, so we have to try both possibilities, i.e. using pairs $([r_1, r_2], [r_3, r_4], ...)$, and pairs $([r_2, r_3], [r_4, r_5], ...)$, respectively. The results for the seven rainfall data series analyzed show a rather remarkable common feature: they all exhibit an autocorrelation of weights $\rho_w(2) \approx -0.2$, corresponding to $\rho_w(1) \approx -0.4$, within a close range. (The values closer to -0.2 for each rainfall event are -0.204, -0.201, -0.2, -0.199, -0.202, -0.203, -0.185; the other values are -0.163, -0.186, -0.191, -0.234, -0.194, -0.223, -0.226.) Although valid only in the context of the somewhat restrictive model of complementary weights, we believe this to be a very interesting universal property. Once established, it allows modeling these rainfalls using binary cascades with complementary weight

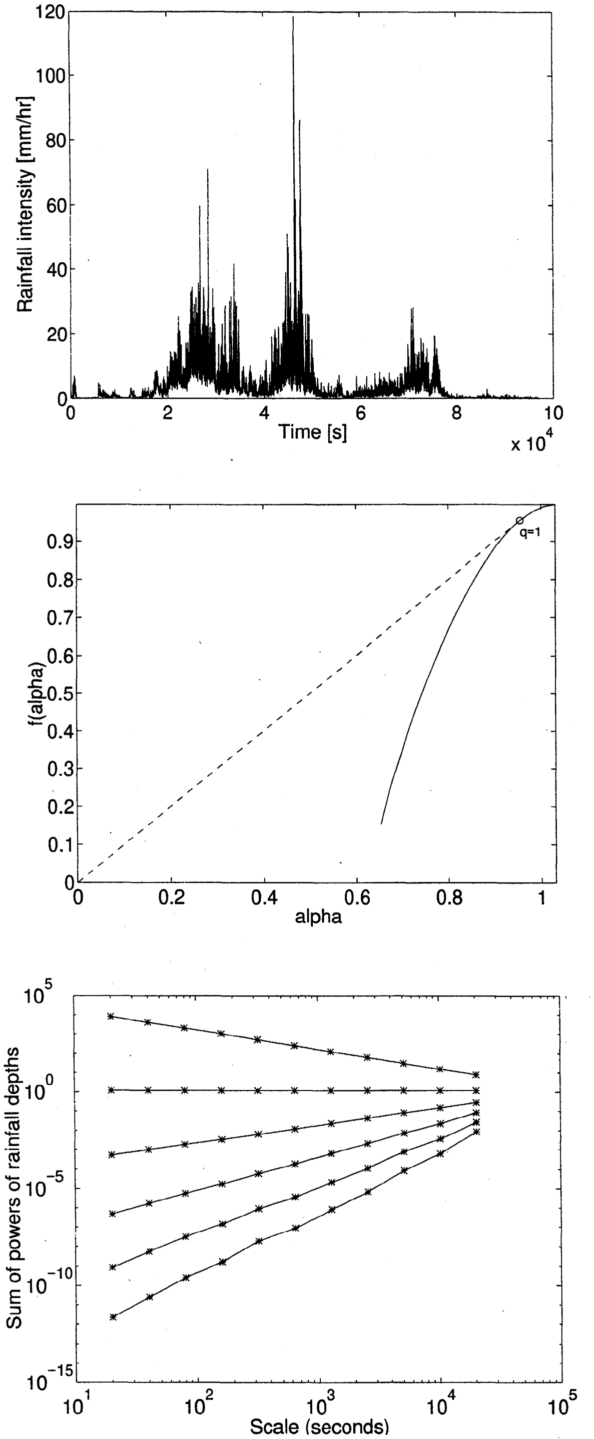

Figure 1. (top) The time series of the rainfall event of December 2, 1990; (middle) the f(α) curve for that event; (bottom) log-log fits of $\sum_k |r_k(\lambda\Delta t)|^q$ against scale $\lambda\Delta t$ for the same event and for $q = 0, 1, 2, 3, 4, 5$.

generators, whose every second weight is drawn from an AR(1) model with lag 1 autocorrelation coefficient -0.2 and a cumulative distribution function (CDF) $F(x) = \int_0^\infty \mathcal{F}(x\chi/(1-x))d\mathcal{F}(\chi)$, where $\mathcal{F}$ is the cumulative probability distribution of the rainfall values. Weight distributions for the analyzed rainfalls have been found to fit fairly well a function that corresponds to a lognormal F. Notice here that since binomial generators are a particular case of complementary weights generators, we can assess from the distribution of weights whether it is appropriate to use a binomial multiplicative cascade for rainfall. To choose a binomial generator, one would like to see a density distribution of weights showing two symmetric spikes, or at least a bimodal distribution, which was not found to be the case for temporal rainfall.

In a more general context than cascades with complementary weights, we will show below that the negative autocorrelation $\rho_w(2)$ carries a physical significance, all the more important as we show that it appears in different types of cascade models for temporal rainfall. Also, for the particular case of binary multiplicative cascades with complementary weights, it is significant to see that the $\rho_w(2)$ values for the seven events analyzed are remarkably close to each other, which reinforces the idea of a common mechanism giving rise to them.

2.2. Indirect Estimation Through Patterns of Variation Preserved Under Aggregation: The Oscillation Coefficients

We would like to relax the requirement of complementarity in weights and have mass conservation only in an average sense across the cascade, since there is no way to assert that our sampling occurs precisely at the same points where the underlying cascade exhibits its branching. However, in doing this, we lose the ability of reconstructing the weights from the rainfall series. In this case, a straightforward (though indirect) way of assessing the dependence in the weights of a multiplicative cascade would be to relate that dependence to the autocorrelation function (ACF) of the cascade model, which can then be compared directly with the ACF of the rainfall series. This, however, is in most cases not possible, since the theoretical ACF of some cascade models might not be defined at all [see also *Holley and Waymire*, 1992]. An issue therefore arises to find a descriptor which is properly defined for most (or all) cascades, is aggregation-invariant for all cascades for which it is defined, and has a distinctive behavior for other processes. The second requirement is needed since there is no criterion to choose a priori one scale of the cascade over another in order to describe a natural phenomenon.

We propose a descriptor of data series that fulfills the above stated requirements, for the purpose of testing model appropriateness and improving/completing cascade model fits. We found such a descriptor to be the series of coefficients which measure the fraction of the total n-tuples in the series that are obeying a certain pattern up-down-up etc. For the sake of intuitiveness, we will denote the coefficients for pairs as $C_\downarrow$ ($= \mathcal{P}[r_k > r_{k+1}]$) and $C_\uparrow$ ($= \mathcal{P}[r_k < r_{k+1}]$), the ones for triplets $C_{\downarrow\downarrow}$, $C_{\uparrow\downarrow}$, $C_{\downarrow\uparrow}$ and $C_{\uparrow\uparrow}$, and so on.

Oscillation coefficients have a number of properties that make them attractive for inferring scale invariance and discriminating between models. Some of these properties are as follows.

Scale invariance in a multiplicative cascade. Oscillation coefficients are scale-invariant for a multiplicative cascade, and appear to be a direct indicator of the dependence structure of weights. In binary cascades the oscillation coefficients turn out not to depend on the weights distribution, as our simulations show. Also, in section 2.3 it is pointed out that $C_{\downarrow\uparrow}+C_{\uparrow\downarrow}$ (hereafter denoted $C_{\updownarrow}$) is in a one-to-one correspondence with the autocorrelation $\rho_w(2)$ in the cascade generator's weights. Therefore the distribution parameters of the generator's weights can be independently chosen to match the spectrum of scaling exponents f(α), whereas the weight dependence structure to match $C_{\updownarrow} \equiv C_{\downarrow\uparrow} + C_{\uparrow\downarrow}$. This allows for uncoupled estimation of the distribution function and the dependence structure of the weights of the cascade generator, emphasizing the importance of taking advantage of this extra degree of freedom when modeling rainfall series with cascade models.

Independence from CDF. For a sequence of independent, identically distributed (i.i.d.) random variables, the coefficients C converge to unique values, independently of the cumulative distribution function F of the process, if F is continuous. It can be shown that $C_\downarrow = 1 - C_\uparrow = 1/2$, $C_{\downarrow\downarrow} = C_{\uparrow\uparrow} = 1/2 - C_{\uparrow\downarrow} = 1/2 - C_{\downarrow\uparrow} = 1/6$, etc., since $C_\downarrow = \int_0^1 F dF$, $C_{\downarrow\downarrow} = \int_0^1 \left(\int_0^F \mathcal{F} d\mathcal{F}\right) dF$, etc. This allows us to discriminate between i.i.d. random processes and many nonlinear processes, something that cannot be inferred from the autocorrelation function which only measures linear correlation. For instance, in the case of a logistic series $x_n = 4x_{n-1}(1 - x_{n-1})$ with $x_0 \in (0, 1)$ neither periodic, nor eventually periodic, the expected values of first-order oscillation coefficients can be shown to be $C_\downarrow = 1 - C_\uparrow = 1/3$ (see Appendix B, section B1), which immediately shows a striking difference from the oscillation coefficients of i.i.d. random variables. Notice also that we can evaluate the C coefficients for other types of models. For instance, a stochastic AR(1) model $x_n = \rho x_{n-1} + \sqrt{1-\rho^2}\epsilon_n$, with lag 1 autocorrelation ρ and with the noise ϵ_n (of cumulative distribution F) independent of x_i and ϵ_i $\forall i = 1, ..., n$-1, has $C_\downarrow = 1 - C_\uparrow = \int_0^1 F\left(x\sqrt{(1-\rho)/(1+\rho)}\right) dF(x)$. If the (limiting) density distribution of the model is an even function (which is often the case), then it can be shown (see Appendix B, section B2) that $C_\downarrow = C_\uparrow = 1/2$. This fact makes first-order oscillation coefficients inappropriate for distinguishing AR(1) models from i.i.d. random variables. Second-order coefficients are, in this case, computed as

$$C_{\downarrow\downarrow} = \mathcal{P}\left[\left[\epsilon_{n+1} < x_n\sqrt{\tfrac{1-\rho}{1+\rho}}\right] \wedge \left[\epsilon_n < x_{n-1}\sqrt{\tfrac{1-\rho}{1+\rho}}\right]\right] =$$
$$\int_0^1 \int_0^{F\left(x\sqrt{\frac{1-\rho}{1+\rho}}\right)} \mathcal{F}\left(\rho x\sqrt{\tfrac{1-\rho}{1+\rho}} + (1-\rho)\chi\right) d\mathcal{F}(\chi) dF(x),$$

and are in general different from those of i.i.d. random variables. Also, for these processes that are not scale-invariant, the oscillation coefficients are not aggregation-invariant either. Rather, their variation over scales depicts the lack of scale-invariance in the process.

Capability of indicating underlying continuity/differentiability. The behavior of the oscillation coefficients for a "closely" sampled differentiable function is easy to understand, and hereby, inferences can be made as to whether the data series stems from such a function, or rather from an inherently discrete process. To illustrate this, we consider a sinusoid signal, sampled at a rate of 128 samples/period, for a total length of eight periods. Aggregating the signal five times by a factor of 2 brings it down to 4 samples/period. Over this whole range of sampling rates, $C_{\updownarrow} \equiv C_{\downarrow\uparrow} + C_{\uparrow\downarrow}$ "feels" the signal as differentiable, by doubling over each aggregation (see Figure 2, inclined solid line). Notice here that the discriminatory power of $C_{\updownarrow}$ resides in depicting whether a signal "is behaving like a differentiable function" over the range of scales analyzed, in the sense of generally conserving its extrema over scales. It therefore gives an indication of whether it makes sense to instead use a discrete, or else a continuous/differentiable model over these scales, but it cannot possibly allow inferences on the nature of the process at scales below the sampling scale, if such scales are at all relevant to the process. As far as continuity is concerned, inferences can be made as to the differentiability of the integral process.

Capability of indicating the presence of noise. The addition of white noise of low amplitude in the above sinusoid (standard deviation of noise is 0.003 of sine amplitude) is felt in $C_{\updownarrow}$ only at small scales (dotted line in Figure 2), afterward it joins the solid line, as ex-

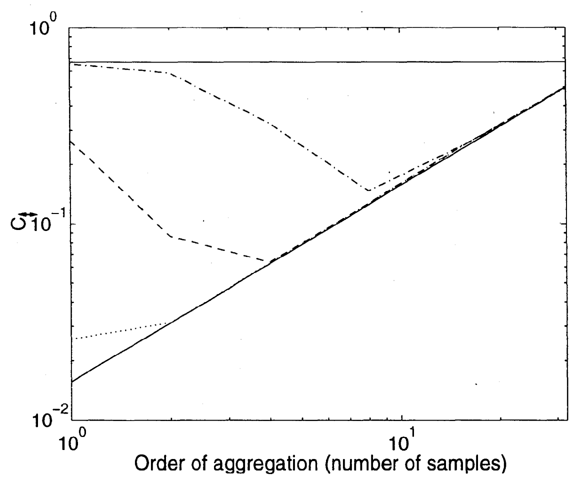

Figure 2. $C_{\updownarrow}$ over repeated aggregation of a sine signal with added white noise of standard deviations 0.003 (dotted line), 0.03 (dashed line), and 0.3 (dash-dotted line) of the sine wave's amplitude. The inclined solid line represents the values of $C_{\updownarrow}$ for the sine wave without noise, and the horizontal solid line represents the theoretical value $C_{\updownarrow} = 2/3$ for a series of i.i.d. random variables.

pected. As we increase the amplitude of the noise (0.03 of the sine amplitude), $C_{\updownarrow}$ becomes more sensitive to the added noise: we see the straight line to very definitely break. Finally, strong noise (0.3 of the sine amplitude) behaves as a sequence of independent random variables at small scales ($C_{\updownarrow} \approx 2/3$ and approximately constant over one aggregation) and joins the solid curve only after four aggregations, when that noise smooths out (see dash-dotted line in Figure 2). Thus the $C_{\updownarrow}$ coefficient can also be used to make inferences about the presence of noise in a smooth signal.

2.3. Behavior of Oscillation Coefficients of a Binary Multiplicative Cascade

Here we try to assess the behavior of oscillation coefficients as we put a dependence structure along the series of weights, at each step of the cascade generation. Although we do not exclude the theoretical possibility of other types of dependence, or of nonstationarity and independence, we use as a measure of dependence among consecutive pairs of weights, the lag 2 autocorrelation of weights $\rho_w(2)$ along one level of the cascade generator. We then try to establish a relationship between $C_{\updownarrow}$ and $\rho_w(2)$. Since an analytical solution would be extremely cumbersome, we establish it by simulation for a binary multiplicative cascade with random, correlated weights positioning. In our simulation, every weight is related to the corresponding weight of an adjacent pair through an autoregressive process. The results indicate that the $C_{\updownarrow} \propto \rho_w(2)$ relationship shown in Figure 3 is independent of the distribution of weights, and is a one-to-one, linearly increasing function. For finite-length series the

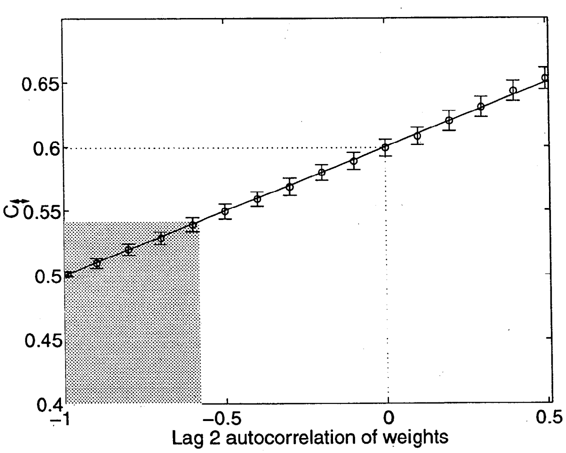

Figure 3. $C_{\updownarrow}$ versus lag 2 autocorrelation coefficient of weights $\rho_w(2)$ in a binary multiplicative cascade. The error bars are shown for simulations with lognormally distributed weights, but the linear relationship itself was found to be independent of the weight distribution. The shaded area corresponds to the region of oscillation coefficients found from the seven analyzed high-resolution temporal rainfall series. For a cascade generator with independent weights ($\rho_w(2) = 0$) the $C_{\updownarrow}$ value was found equal to 0.6 by simulation (dotted line).

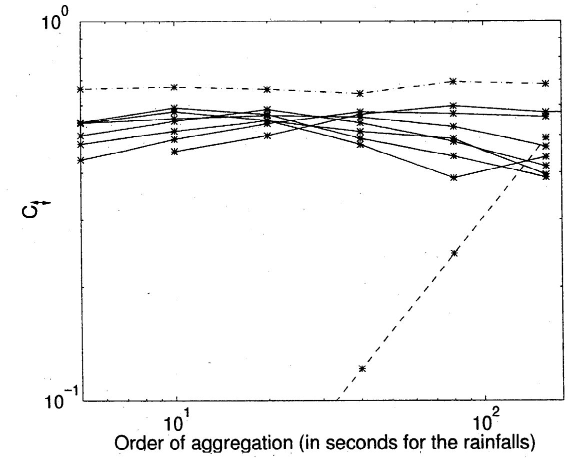

Figure 4. $C_{\updownarrow}$ versus aggregation length in the seven rainfall time series (solid lines). The dash-dotted line represents a realization for a series of i.i.d. random variables of a length comparable to the rainfall series (theoretical value of 2/3 shown dotted in Figure 2); the dashed line stands for a smooth signal (upper part of the inclined solid line in Figure 2).

realized values of $C_{\updownarrow}$ have been found to have a Gaussian spread at any given $\rho_w(2)$, not surprising as the C coefficients can be regarded as average measures of oscillation patterns along a series and are expected to obey the central limit theorem. The error bars in Figure 3 indicate plus/minus one standard deviation, denoted σ_{4096} for a series of 4096 values (a length of the order of magnitude of the rainfall time series). The standard deviation values of Figure 3 can be used for a sequence of arbitrary length l by setting $\sigma_l = \sigma_{4096}\sqrt{4096/l}$.

3. Matching the Oscillation Coefficients of a Cascade Model for Temporal Rainfall

Study of the oscillation coefficients of the seven temporal rainfall series indicates the following:

1. $C_{\updownarrow}$ is indeed invariant over all analyzed scales, i.e., between the 5-s scale of sampling and scales of the order of magnitude of the storm length, of several hours (see Figure 4). This reinforces the hypothesis that these rainfall time series show scale invariance. (Notice that the observed increase in the spread of the $C_{\updownarrow}$ estimates over aggregation is statistically expected, as discussed in the previous section.) At the same time, invariance of (a nonzero) $C_{\updownarrow}$ over scales shows that the process does not exhibit a differentiable function behavior at any scale between 5 seconds and storm duration (not even at the smallest scale of 5 s, below which the discrete character of droplets renders the process inherently discontinuous).

2. Although aggregation-invariant, the $C_{\updownarrow}$ coefficients for the rainfall time series (Figure 4) are quite far from the 2/3 value of i.i.d. random variables, considering any reasonable probability bounds (as all but one of the time series have more than 4096 values, and

the standard deviation of $C_{\updownarrow}$ around 2/3 at 4096 values is $\approx 6.7 \times 10^{-3}$). This practically excludes an i.i.d. random variables hypothesis for rainfall. It also excludes the possibility of strong added noise.

3. Finally, and most important, when the values of the $C_{\updownarrow}$ coefficients for the rainfall time series (see Figure 4) are compared to those of a binary multiplicative cascade, they do not match the hypothesis of independent weights in the cascade generator. Rather, they match the zone of high negative correlation $\rho_w(2)$, as can be seen from Figure 3.

Since in this case too, similar to the case of complementary weights, we obtain consistently that $\rho_w(2)$ should be negative for temporal rainfall models, we feel that there must be a physical cause to this fact. Under the hypothesis that local rainfall intensities are influenced by turbulent eddies of different scales, and that within scales of a few minutes atmospheric structures are convected over a fixed location, let us try to analyze whether and how it is likely to have this particular weight dependence structure in place. Conceivably, both the interactions between adjacent eddies at the same scale, as well as between eddies decaying from large scales to small scales, are in play. In our inverse problem of determining the dependence structure of weight positioning for a multiplicative cascade rainfall model we only employed correlation along the same level of the cascade generator. Notice however that in order to explain how this dependence structure appears in real-life data, it is necessary to look at the physical interactions of both types, that is, at the same scale as well as along scales.

In a study of how turbulent eddies translate into the multiplicative cascade, *Arnéodo et al.* [1992], show that the branching structure in the time-frequency plane of a turbulent wind tunnel signal keeps the same symmetries over all scales present. As far as precipitation is concerned, eddies rolling parallel to the Earth's surface would tend to create asymmetries in the quantities of rainfall passing through the ascending and the descending sides of the eddy, respectively, as an effect of entrainment and coalescence of drops. Therefore the values of weights induced by each eddy would correspond to its turning direction. In this context, the interaction between eddies of the same scale, having a tendency to roll against each other, translates (via a frozen-field hypothesis) rather simply into negative correlation between adjacent pairs of generator weights, at every level of the cascade. This observation, by itself, would be enough to account for the negative correlation $\rho_w(2)$ in weights found herein, but we know for a fact that interactions among eddies at different scales do also exist, and these interactions seem more difficult to pinpoint than those of adjacent eddies at the same scale, and even more so, to translate in terms of cascade weights positioning. Let us therefore look at a general form of (statistical) dependence between weights at two successive levels in a binary multiplicative cascade model with complementary weights. We consider a correlation ρ between a weight at a certain level of the cascade (corresponding to a certain scale in the process) and one of the weights into which it subsequently splits. In Appendix C we show that this dependence between different levels of the cascade induces a dependence along each level of the cascade, which in terms of $\rho_w(2)$ is

$$\rho_w(2) = -\frac{\rho^2}{2+\rho^2} \qquad (2)$$

The curve obtained for $\rho_w(2)$ as a function of ρ is shown in Figure 5. The remarkable fact is that all the values of $\rho_w(2)$ are negative, i.e., we may conclude that any correlation from one level of the cascade to the next results in a negative correlation $\rho_w(2)$. (Independence between cascade levels yields as expected a zero correlation $\rho_w(2)$.) This fact, together with the considerations regarding the interaction of eddies at the same scale, tells us that a cascade model with negative correlation of weights pairs, as obtained from the analysis of rainfall time series, also makes full sense from the point of view of the heuristics of the physical phenomenon. Moreover, the relatively narrow ranges of $\rho_w(2)$ values for all seven rainfall series of different seasons and durations (see Figure 3) corroborate the importance of looking further into this antisymmetric energy-cascading mechanism as a unique underlying property of rainfall.

4. Conclusions

In this paper the need for exploring dependence structures in the weights of a multiplicative cascade model for temporal rainfall was argued for. A quantity $C_{\updownarrow}$ based on "oscillation coefficients" was introduced and it was shown that it is scale-invariant for a cascade model and that it has the ability to depict the presence and

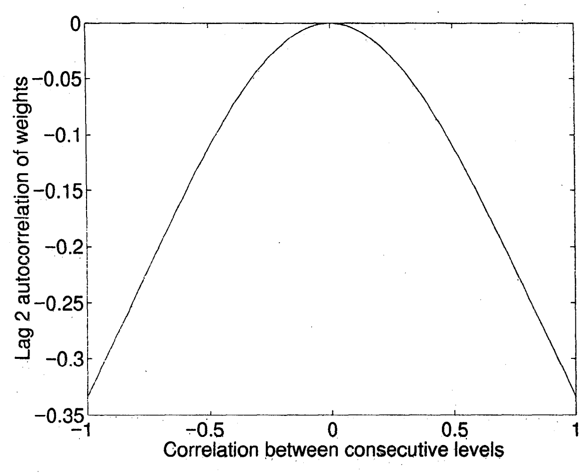

Figure 5. The limiting value of the autocorrelation $\rho_w(2)$ in a binary multiplicative cascade with complementary weights (see equation 2), resulting from the dependence between successive levels of the cascade, as a function of the correlation ρ (defined in the text) between weights at consecutive levels.

type of correlation in the weights of a multiplicative cascade generator.

Application of the above developments to seven high-resolution temporal rainfall series has indicated the following. (1) Invariance of $C_{\updownarrow} = C_{\uparrow\downarrow} + C_{\downarrow\uparrow}$ over all existing scales (between the 5-second scale of sampling and scales of the order of magnitude of the storm length, i.e., several hours) confirms the aggregation-invariance of the rainfall process as shown by multifractal analysis, and also firmly indicates that these rainfall time series do not exhibit differentiability at any of these scales. (2) Although aggregation-invariant, $C_{\updownarrow}$ is quite far from the 2/3 value of i.i.d. random variables, considering any reasonable probability bounds, which practically excludes the i.i.d. random variables hypothesis, and therefore the hypothesis of aggregation invariance achieved by Lévy-stably (including normally) distributed, independent random variables. (3) A zone of negative correlation between consecutive pairs of weights in a binary multiplicative cascade resulted both directly (in the case of complementary weights), as well as indirectly through the oscillation coefficients (in the case of noncomplementary weights).

It must be emphasized that in this work, we examine two types of multiplicative cascade generators, which in fact are "extremes" as far as the correlation within the pair of generator weights is concerned. This correlation is by default equal to -1 in the case of the complementary weights $[w, 1-w]$ generator, and was set here to 0 in the case of the noncomplementary weights generator. However, in the case of the noncomplementary weights generator, the correlation of weights within a pair could be set to a nonzero value, adding thus one more degree of freedom (and one more parameter) in the cascade generator. This correlation within the pair of weights is heuristically expected to be of some negative value, since local mass conservation would have an effect in a statistical sense in the "split" of the cascading values. Therefore, in the future, resolving this additional parameter of the cascades used for modeling temporal rainfall would offer a more complete model and would also establish a contiguous domain for the values of $\rho_w(2)$.

At the present time, the match of the temporal rainfall data series with a binary multiplicative cascade with consistently negative correlation between adjacent pairs of weights along the sequence can be, as shown, interpreted as a match with a cascade exhibiting dependence between one scale to the next. This is important, in our opinion, since it appears that bifurcation patterns of rainfall shown in the time-frequency domain exhibit direct or inverse symmetries along successive scales [e.g., *Venugopal and Foufoula-Georgiou*, 1996]. This kind of dependence, otherwise difficult to depict in real-life data for reasons of inherent corruption by side effects, is in our opinion not only providing a better fit of the dependent-weight multiplicative cascade model to temporal rainfall event data, but is also a step forward in understanding the phenomenon.

Appendix A: Autocorrelation Coefficients of Weights in a Binary Multiplicative Cascade With Complementary Weights

Here we show that, for a binary multiplicative cascade with complementary weights $[w, 1-w]$, $\rho_w(1)$ relates linearly to $\rho_w(2)$. Denoting by E the expected value operator, the lag 1 autocorrelation of the weights sequence expressed by averaging over the two types of consecutive weights that appear, i.e., within a pair $[w, 1-w]$ and in between pairs, is

$$\begin{aligned}\rho_w(1) &= \frac{E_i\{(w_i-1/2)(1-w_i-1/2)\}}{2E_i\{(w_i-1/2)^2\}} + \\ &\quad \frac{E_i\{(1-w_i-1/2)(w_{i+2}-1/2)\}}{2E_i\{(w_i-1/2)^2\}} \\ &= -\frac{E_i\{(w_i-1/2)^2\}}{2E_i\{(w_i-1/2)^2\}} - \\ &\quad \frac{E_i\{(w_i-1/2)(w_{i+2}-1/2)\}}{2E_i\{(w_i-1/2)^2\}} \\ &= -\frac{1+\rho_w(2)}{2}.\end{aligned}$$

Appendix B: Derivation of Oscillation Coefficients

B1. Logistic Function

The invariant cumulative distribution for the nonperiodic orbits of the logistic function $x_n = 4x_{n-1}(1 - x_{n-1})$ with $x_0 \in (0,1)$ (no attracting periodic orbits exist) is $F(x) = \int_0^x \pi^{-1}\chi^{-1/2}(1-\chi)^{-1/2}d\chi = 1/2 + \arcsin(2x-1)/\pi$. Fixed points are given by $x^* = 4x^*(1-x^*) \iff x^*(3-4x^*) = 0$, so $x_1^* = 0$ and $x_2^* = 3/4$. It turns out that $x_n > x_{n-1}$ for $0 < x_{n-1} < 3/4$, and $x_n \leq x_{n-1}$ otherwise. We therefore have $C_{\uparrow} = \mathcal{P}[0 < x < 3/4]$, and in turn $C_{\downarrow} = 1 - C_{\uparrow} = 1 - F(3/4) = 1/2 - \arcsin(1/2)/\pi = 1/3$.

B2. AR(1) Model

In an AR(1) model $x_n = \rho x_{n-1} + \sqrt{1-\rho^2}\epsilon_n$, with lag 1 autocorrelation ρ and with the noise ϵ_n (of cumulative distribution F) independent of x_i and ϵ_i, $\forall i = 1, ..., n\text{-}1$, if the distribution of the noise is a stable distribution (attractive under aggregation), then x_n will have that same stable distribution. Moreover, if the density distribution is an even function (which is often the case), then $F(x) - F(0)$ is an odd function, and so is $F(ax) - F(0)$, $\forall a \in \mathbf{R}$. That implies $\int_0^1 (F(ax) - F(0))dF(x) = 0$, and therefore $\int_0^1 F(ax)dF(x) = F(0)\int_0^1 dF(x) \equiv F(0)$. Since $F(x) - F(0)$ is an odd function, and $F(-\infty) = 0$, $F(\infty) = 1$, we have $F(0) = 1/2$ and so $C_{\downarrow} = C_{\uparrow} = 1/2$. A rather simple heuristic of symmetry tells us that the same is true for all ARMA models with symmetric distributions.

Appendix C: Derivation of $\rho_w(2)$ as a Function of the Correlation ρ Between Weights at Consecutive Levels in a Binary Multiplicative Cascade With Complementary Weights

In order to express $\rho_w(2)$ as a function of ρ, we look at its behavior as the cascade splits from larger scales to smaller scales, and try to find a limiting value. If we adopt an AR(1) model to describe the autocorrelation from a larger scale (k) to a smaller scale ($k+1$), we have:

$$w_{2i}^{(k+1)} - 1/2 = (w_i^{(k)} - 1/2)\rho + \epsilon_i$$

and

$$w_{2i+2}^{(k+1)} - 1/2 = (w_{i+1}^{(k)} - 1/2)\rho + \epsilon_{i+1},$$

which leads to

$$E_i\{(w_{2i}^{(k+1)} - 1/2)(w_{2i+2}^{(k+1)} - 1/2)\} =$$

$$\rho^2 E_i\{(w_i^{(k)} - 1/2)(w_{i+1}^{(k)} - 1/2)\}$$

$$\Longrightarrow \rho_w^{(k+1)}(2) = \rho^2 \rho_w^{(k)}(1) \overset{\S A}{=} -\rho^2 \frac{\rho_w^{(k)}(2) + 1}{2}.$$

The fixed point of $\rho_w(2)$ consequently results:

$$\rho_w^*(2) = -\rho^2 \frac{\rho_w^*(2) + 1}{2} \Longrightarrow \rho_w^*(2) = -\frac{\rho^2}{2 + \rho^2}.$$

This fixed point is indeed attracting, since the absolute value of the derivative $\rho^2/2$ of the iterated function is strictly less than unity. Therefore the above expression is always the limiting value of $\rho_w(2)$ in a binary multiplicative cascade with complementary weights, for all $\rho \in [-1, 1]$.

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