Subordinated Brownian motion model for sediment transport

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Following the introduction of the Brownian motion model for sediment transport by Einstein, several stochastic models have been explored in the literature motivated by the need to reproduce the observed non-Gaussian probability density functions (PDFs) of the sediment transport rates observed in laboratory experiments. Recent studies have presented evidence that PDFs of bed elevation and sediment transport rates depend on time scale (sampling time), but this dependence is not accounted for in any previous stochastic models. Here we propose an extension of Brownian motion, called fractional Laplace motion, as a model for sediment transport which acknowledges the fact that the time over which the gravel particles are in motion is in itself a random variable. We show that this model reproduces the multiscale statistics of sediment transport rates as quantified via a large-scale laboratory experiment.

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I. INTRODUCTION

Stochastic theories of sediment transport were initiated with the seminal work of Einstein [1], who introduced a Brownian motion model for particle motion. Since then, these theories were advanced by the need to reproduce the observed statistics of sediment transport rates or particle movement. In [2], a birth-death process was proposed for sediment transport, which was later shown, in [3], to be inadequate as it failed to predict the heavy tails found in the probability density functions (PDFs) of the number of moving particles in a given observation window. In [4], the birth-death model was extended to a birth-death-immigration-emigration model to reproduce the experimentally observed negative binomial distributions for the number of moving sediment particles. The stochastic nature of sediment particle entrainment has been widely recognized and considerable efforts have been invested in modeling this behavior [5–8]. The underlying assumption of these models is that the shear stress, which is the initiator for sediment entrainment, follows a Gaussian distribution. However, many experimental studies have shown that the shear stress fluctuations do not follow a Gaussian distribution, and in particular it has been shown that they follow a Gamma distribution (e.g., [9,10]). The role of near-bed turbulence in sediment transport has also been recognized to play an important role [11,12]. However, turbulence is well known to exhibit variability over a range of scales, and it is reasonable to ask whether this multiscale variability shows its effect on sediment transport series and bed elevation fluctuations.

In a recent study [13], the dependence of the statistics of sediment transport on time scale (sampling time) akin to the scale-dependent statistics of fully developed turbulence [14] was documented. Specifically, it was shown that the PDF of sediment transport rates at small sampling times exhibits a heavy-tailed distribution which however approaches a Gaussian distribution as the sampling time increases. To the best of our knowledge, no stochastic model of sediment transport exists which reproduces this observed multiscale statistical structure of sediment transport series. It is the scope of this paper to present such a model and discuss its mathematical properties and its physical relevance to modeling sediment transport.

The paper is structured as follows. In the following section a brief review of multiscale statistics of sediment transport series observed in a large-scale laboratory experiment is given. In Sec. III the application of a stochastic model, called the fractional Laplace motion, is proposed to characterize the sediment transport series and is shown that it is able to reproduce the observed statistics. In Sec. IV the proposed model is validated against the sediment transport series obtained from a large-scale laboratory experiment. Finally, discussion and conclusions are given in Secs. V and VI.

II. MULTISCALE STATISTICS OF SEDIMENT TRANSPORT SERIES

A large-scale laboratory experiment was recently conducted in the Main Channel facility at the St. Anthony Falls Laboratory, University of Minnesota, in order to study sediment transport dynamics in gravel and sand-bed rivers. The details of the experimental facility can be found in [13,15,16]. Here we briefly describe one of the experiments from which data were used in this study. The flume is 2.74 m wide and 55 m long, with a maximum depth of 1.8 m (see Fig. 1). Gravel with a median particle size (D50) of 11.3 mm was placed in a 20-m-long mobile-bed section of the 55-m-long channel. A constant discharge of water at 4300 liters per second was released into the flume. At the downstream end

FIG. 1. Experimental flume facility at the St. Anthony Falls Laboratory, University of Minnesota.

of the test section was located a bedload trap, consisting of five weighing pans of equal size that spanned the width of the channel. Any bedload sediment transported to the end of the test section of the channel would fall into the pans, which automatically recorded the weight of the accumulated sediment every 1.1 s. Data were collected over a period of 30 h once a state of statistical equilibrium was reached (see [13,16]). The original series of 1.1 s sampling interval were converted to 2 min sediment accumulations via moving averaging in order to remove mechanical (due to vibration) noise present in the raw data (see [13,16]). Let us denote by $S(t)$ the 2 min sediment accumulation series which is shown in Fig. 2. In this section, we present the multiscale analysis performed on this sediment transport series.

The goal of a multiscale analysis is to quantify the manner in which the statistics of the local fluctuations, or variability in a series, changes with scale. In order to investigate the multiscale structure of $S(t)$ over a range of scales, differences (or increments) were computed at different scales (lags) r, denoted by $\delta S(t,r)$, as

$$\delta S(t,r) = S(t + r\Delta t) - S(t), \tag{1}$$

i.e., $\delta S(t,r)$ is the incremental sediment accumulation within a time interval $r\Delta t$, where $\Delta t = 2$ mins. In [13], "generalized fluctuations" were used defined via wavelet transforms (acting as a differencing filter). Notice that while $S(t)$ can only be positive, the fluctuation series $\delta S(t,r)$ will have zero mean and can be both positive and negative. The estimates of the qth-order statistical moments of the absolute values of sediment transport increments at scale r, also called the partition functions or structure functions, $M(q,r)$ are defined as

$$M(q,r) = \frac{1}{N_r} \sum_{t=1}^{N_r} |\delta S(t,r)|^q, \tag{2}$$

where N_r is the number of data points of sediment transport increments at a scale r. The statistical moments $M(q,r)$ for all q completely describe the shape of the PDFs as the scale r changes. Statistical scaling, or scale invariance, requires that $M(q,r)$ is a power-law function of the scale, that is,

$$M(q,r) \sim r^{\tau(q)}, \tag{3}$$

where $\tau(q)$ is the so-called scaling exponent function. For a scale-invariant series, it has been shown that the function $\tau(q)$ completely determines how the PDF of the variable changes with scale (e.g., [17,18]). The simplest form of scaling, known as simple scaling or monoscaling, is when the scaling exponents are a linear function of the moment order, i.e., when $\tau(q) = Hq$. In this case, the shape of the PDF remains the same over scales apart from a rescaling by a deterministic function which depends on the single parameter H. If $\tau(q)$ is nonlinear, the shape of the PDF changes over scales and more than one parameter is required to describe this change (e.g., [17,18]). In this case, the series is called a multifractal. For most processes the nonlinear relationship of $\tau(q)$ with q can be parameterized as a polynomial, and the simplest form is a quadratic approximation,

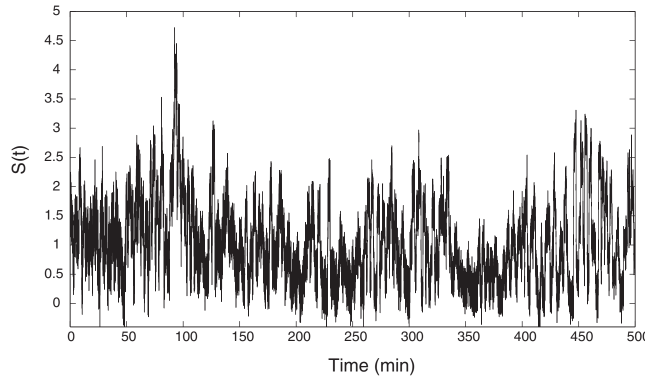

FIG. 2. Sediment transport series $S(t)$ (in kgs) at a sampling interval of 2 min, i.e., series of 2 min sediment accumulation.

$$\tau(q) = c_1 q - \frac{c_2}{2} q^2. \tag{4}$$

The multiscale analysis in this framework provides a compact way, using two parameters c_1 and c_2, of parametrizing the change of the PDF over a range of scales. In parallel to the statistical interpretation of these parameters, there is also a geometrical interpretation. Specifically, the parameter c_1 is a measure of the average "roughness" of the series and c_2, the so-called intermittency coefficient, is a measure of the temporal heterogeneity of the abrupt local fluctuations in the series [in fact, it relates to the variance of the so-called local Hölder exponent which measures the local degree of nondifferentiability of the series (e.g., [18])]. It is noted that using a higher than second degree polynomial approximation of $\tau(q)$, say a third degree polynomial, introduces a third parameter c_3, which is a measure of the third moment of the local differentiability of the series and it might be hard to accurately estimate from a limited sample size of data. Thus, in most practical applications the approximation of $\tau(q)$ curve is restricted to a quadratic function which is parameterized by c_1 and c_2. Estimation of the multifractal parameters, c_1 and c_2, can be performed in various ways. For example, one can use a quadratic fit to the whole $\tau(q)$ curve [estimated for several values of q from the slopes of $M(q,r)$ vs r in log-log space] or use the first two scaling exponents only, $\tau(1)$ and $\tau(2)$, or use the cumulant analysis method (e.g., [18] and references therein). In this study, we use the quadratic fit to the $\tau(q)$ curve for the estimation of the parameters c_1 and c_2.

The multiscale analysis described above was performed on the sediment transport series shown in Fig. 2. Figure 3(a) shows the scaling of the moments of the sediment transport increment series $\delta S(t,r)$ with scale r. It is to note that the structure functions follow a power-law relation in r over a range of scales from $r = 4$ to 64 (8 to 128 min). The scaling exponents of the structure functions $M(q,r)$ are plotted as a function of the order of moments q in Fig. 3(b) for $q = 0.5, 1, 1.5, \ldots, 3$. We observe that $\tau(q)$ has a nonlinear dependence on q, which is an indication of the presence of multifractality and the fact that the shape of the PDF changes with scale. Figure 3(c) displays the PDFs of sediment transport increments at two scales, $r = 10$ and $r = 60$ (i.e., 20 and

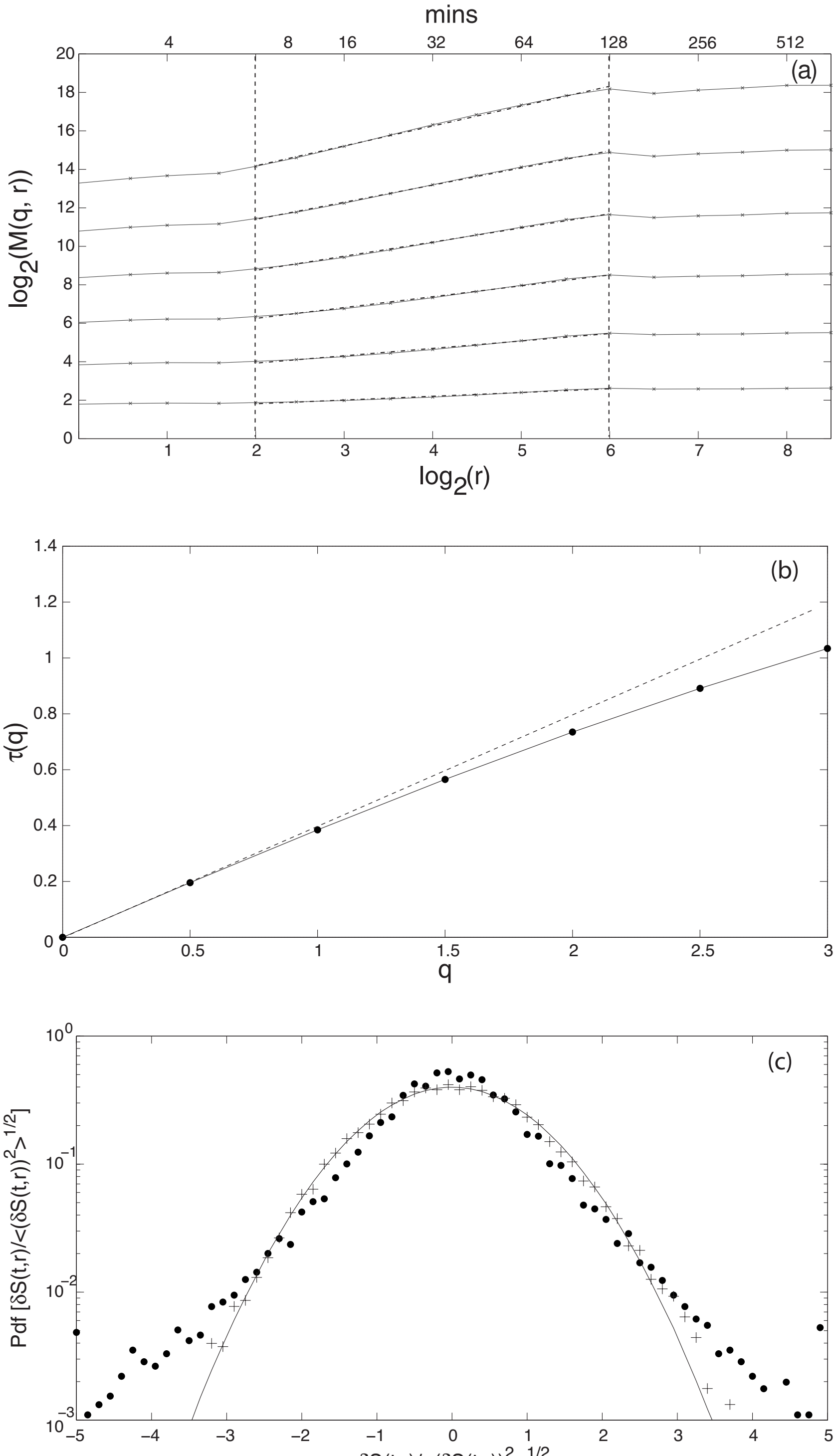

FIG. 3. (a) Structure functions of sediment transport series. Vertical lines delineate the scaling regime which is between 8 and 128 min (see top horizontal axis). (b) Estimated $\tau(q)$ curve (solid points) from the slopes of structure functions and a quadratic fit (solid line). Deviation from the straight line establishes the presence of multifractality (see text for parameter values). (c) Change in PDF of sediment transport increments with scale. The solid dots correspond to PDF at increments of $r=10$ (20 min) and + correspond to increments at $r=60$ (120 min). The solid line indicates a Gaussian PDF.

120 min sediment accumulations, respectively). It is noted that at smaller scales the PDF of the sediment increments deviates from a Gaussian distribution and is close to a double exponential. The PDF, eventually, becomes Gaussian at larger scales. The PDFs reported in Fig. 3(c) are for scales that fall within the scaling regime of the sediment data series [see Fig. 3(a)]. The dependence of the statistics of the sediment transport rates on scale has also been documented in field observations (see [19] and a discussion in [13]). As discussed above, we estimated the parameters of multifractality by approximating the $\tau(q)$ curve in Fig. 3(b) as a quadratic function in q and the estimates obtained together with their 95% standard errors were $c_1=0.41\pm0.005$ and $c_2=0.04\pm0.004$. It is noted for comparison that the c_2 estimate of velocity fluctuations in fully developed turbulence is of the order of 0.03 [14]. We emphasize that no existing stochastic model for sediment transport addresses the issue of statistical scale dependence documented in experimental and field observations. In the following section, we propose a stochastic model for sediment transport which exhibits the

observed change in PDFs of sediment transport increments over scales, reproduces the multifractal behavior of the experimental data series, and provides the potential for relating the observed macroscale statistics to the microscale dynamics of sediment particle movement.

III. PROPOSED MODEL: FRACTIONAL LAPLACE MOTION

A. Brownian motion

Brownian motion is widely recognized to be a special case of a continuous time random walk (CTRW). In general, CTRWs specify the particle location x_i at a time t_i by the iterative discrete equations (e.g., [20,21]),

$$x_{i+1} = x_i + \eta_i, \tag{5a}$$

$$t_{i+1} = t_i + \tau_i, \tag{5b}$$

where (η_i, τ_i) is a set of random numbers drawn from a PDF $\Psi(\eta, \tau)$. One can recast the above equations in the following form:

$$t_n = \sum_{i=1}^{n} \tau_i, \tag{6a}$$

$$x(t) = \sum_{i=1}^{n} \eta_i, \tag{6b}$$

where $t \in [t_n, t_{n+1})$. The CTRW is said to be decoupled when the random variables η_i and τ_i are mutually independent. Brownian motion is a special case of a decoupled CTRW where η_i are independent identically distributed (i. i. d) random variables drawn from a Gaussian distribution and τ_i are i. i. d random variables sampled from an exponential distribution. It is to note that the increments of Brownian motion follow a Gaussian distribution. However, the increments of most natural phenomena often show deviation from Gaussian PDFs and this has prompted the introduction of other stochastic processes such as Lévy walks and continuous-time Lévy flights, where the random variables η_i and/or τ_i are sampled from heavy-tailed PDFs. However, such processes do not have all of their statistical moments convergent. For example, Lévy walks and Lévy flights do not have convergent second moments [22]. It is also noted that modeling real data with such processes typically requires an exponential truncation of the algebraic decays [23] or sometimes even milder than algebraic decay [24]. Correlation and long-range dependence in the observed data can be modeled by relaxing the independence assumption in sampling η_i and/or τ_i or by relaxing the independence assumption of a decoupled CTRW. Fractional Brownian motion, denoted by $B_H(t)$, is a decoupled CTRW starting at zero and has the following correlation function:

$$E[B_H(t)B_H(s)] = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}), \tag{7}$$

where $E(.)$ denotes the expectation operator and H is a parameter of fractional Brownian motion called the Hurst exponent. For $H=0.5$, the fractional Brownian motion reduces to the standard Brownian motion with independent increments. For other values of $0<H<1$, $B_H(t)$ is called the fractional Brownian motion and its increments are positively correlated for $H>0.5$ and negatively correlated for $H<0.5$.

An extension of Brownian motion, or fractional Brownian motion, can be obtained via subordination. The notion of subordination was originated by Bochner [25]. One can obtain a subordinated stochastic process $Y(t)=X[T(t)]$ by randomizing the clock time of a stochastic process $X(t)$ using a new time $t_*=T(t)$. The resulting process $Y(t)$ is said to be subordinated to the so-called *parent process* $X(t_*)$, and t_* is commonly referred to as the *operational time* [26]. We propose the application of subordination of fractional Brownian motion (called fractional Laplace motion) as an extension to the Brownian motion model proposed by Einstein for sediment transport [2]. In the following subsection, we describe the properties of subordinated fractional Brownian motion.

B. Fractional Laplace motion

Fractional Laplace motion is a subordinated stochastic process, whose *parent process* is fractional Brownian motion and the *operational time* is a Gamma process [27],

$$L(t) = B_H(\Gamma_t), \tag{8}$$

where $B_H(t)$ is fractional Brownian motion with Hurst exponent $0<H<1$ and Γ_t represents a Gamma process for any $t\geq 0$. The increments of the Gamma process $(\Gamma_{t+s}-\Gamma_t)$ have a gamma distribution with shape parameter $\nu=s$ and scale parameter $\beta=1$, i.e.,

$$f(x) = \frac{1}{\beta^{\nu}\Gamma(\nu)} x^{\nu-1} e^{-x}. \tag{9}$$

For $H=0.5$ the subordinated process $L(t)=B_H(\Gamma_t)$ is called the Laplace motion.

Increments of the fractional Laplace motion defined by $Y(t,r)=L(t+r)-L(t)$, called the fractional Laplace noise, form a stationary process. Fractional Laplace noise has three parameters, namely, the Hurst exponent of the parent process H, the variance of the parent process $B_H(t)$ at the smallest scale $t=1$, i.e., $\sigma^2=\mathrm{Var}[B_H(1)]$, and the shape parameter of the Gamma process (Γ_t), ν. The variance of the fractional Laplace noise can be expressed as a function of the scale r and its parameters as [27]

$$\mathrm{Var}[Y(t,r)] = \sigma^2 \frac{\Gamma(2H + r/\nu)}{\Gamma(r/\nu)}. \tag{10}$$

The covariance function of the fractional Laplace noise at a given scale r, defined as $\rho(n)=E[Y(t,r)Y(t+n,r)]$, can be expressed in terms of its parameters for any $n\geq 1$ as

$$\rho(n) = \frac{\sigma^2}{2}\left\{\frac{\Gamma[2H+(n+1)r/\nu]}{\Gamma[(n+1)r/\nu]} + \frac{\Gamma[2H+(n-1)r/\nu]}{\Gamma[(n-1)r/\nu]} - 2\frac{\Gamma(2H+nr/\nu)}{[\Gamma(nr/\nu)]}\right\}. \tag{11}$$

Fractional Laplace noise is positively correlated for $H>0.5$

and is negatively correlated for $H<0.5$. In particular, fractional Laplace noise exhibits long-range dependence for $H>0.5$.

The fundamental difference between fractional Laplace motion and other similar stochastic processes such as fractional Brownian motion and Lévy motion is that in the latter two cases the PDFs of the increments remain Gaussian and Lévy stable, respectively, at all scales. In fractional Laplace motion, the PDFs of the increments are variable with scale with Laplace PDFs at small scales and as the scale increases the PDFs approach Gaussian. In particular, fractional Laplace motion deviates from the classical self-similarity and shows stochastic self-similarity [27]. The Laplace PDF emerges from a different and less well-known central limit theorem called the geometric central limit theorem, which states that the sum of a random number of independent identically distributed variates with finite variance is asymptotically Laplace if the random count is geometrically distributed [28]. In fact, the Laplace PDF can be considered as a Gaussian PDF with a random variance or spread [29]. Given the stochastic self-similarity extensively documented in sediment transport series (in [13] and also in Sec. II of this paper), the subordination of the fractional Brownian motion model proposed herein offers an attractive and simple extension to Brownian motion for particle movement, as demonstrated in more detail in the next section.

IV. FRACTIONAL LAPLACE MOTION MODEL FOR SEDIMENT TRANSPORT

The physical relevance of the fractional Laplace motion to model sediment transport is argued on the basis that the notion of operational time acknowledges the randomness in the entrainment time experienced by sediment particles which are subject to a varied range of velocities in turbulent flows. It is known that turbulent velocity fluctuations themselves exhibit intermittency and possess a multifractal behavior (e.g., [14]). Turbulent velocity "sweeps" and "bursts" are expected to influence particle motion and introduce a multiscale variability in the fluctuations of the resulting sediment transport series. In groundwater hydrology, the notion of operational time has been used to acknowledge the fact that time passes faster for particles in higher velocity zones [30,31]. Along these lines, a subordinated Brownian motion model has been proposed to model hydraulic conductivity [28] and connections between turbulent velocities and heterogeneous sediment properties have been proposed [32].

In the following subsections we study the multiscale properties of fractional Laplace motion and show that fractional Laplace motion reproduces the intricate stochastic structure shown by the sediment transport series. Further, we elaborate on the model parameter fitting to the sediment transport series.

A. Multifractal properties of fractional Laplace motion

In order to study the self-similar behavior of fractional Laplace motion, we first study the analytical behavior of the structure functions of fractional Laplace motion. The structure functions of fractional Laplace motion for $\sigma=1$ can be written in terms of its parameters H and ν as [27]

$$M(q,r)=\sqrt{\frac{2^q}{\pi}}\Gamma\left(\frac{1+q}{2}\right)\frac{\Gamma(Hq+r/\nu)}{\Gamma(r/\nu)}. \qquad (12)$$

Statistical scaling or self-similar behavior requires that the structure functions follow a power-law relationship in scales. Figure 4(a) shows the structure-function dependence on

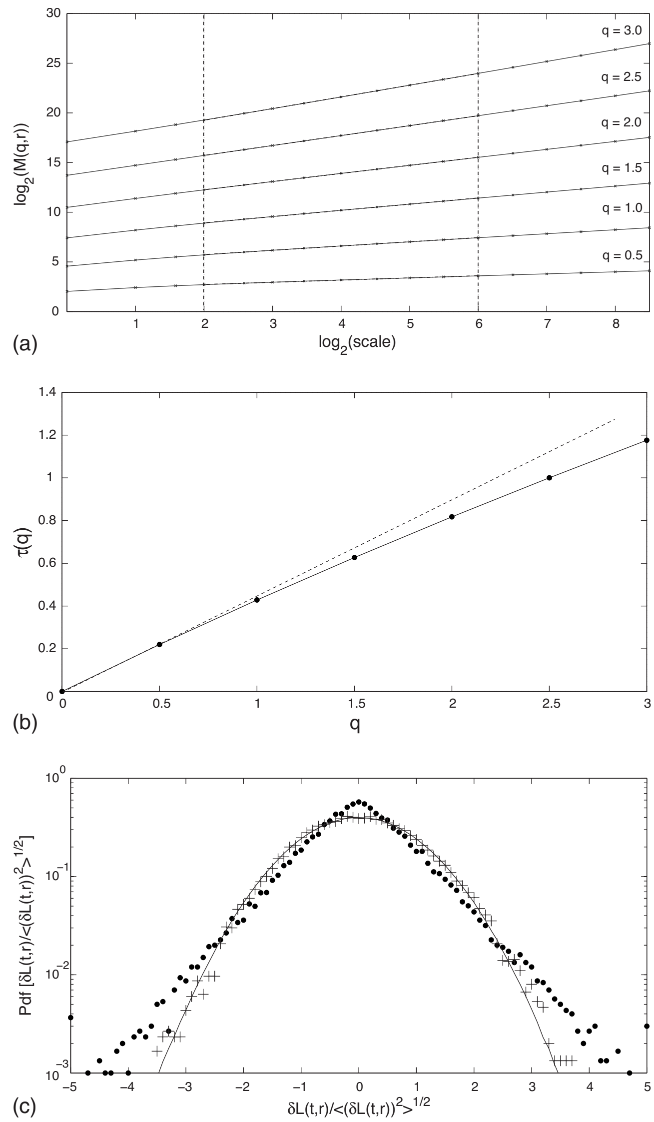

FIG. 4. (a) Structure functions of fractional Laplace motion for a set of chosen parameters $H=0.4$, $\nu=3.0$, and $\sigma=1$ computed from Eq. (12). The vertical lines correspond to the scaling regime of the sediment transport series which is from scales of $r=4$ to $r=64$. (b) Estimated $\tau(q)$ curve (solid points) from the fitted slopes of the structure functions. The solid line indicates a quadratic fit and the nonlinear dependence of $\tau(q)$ on q establishes that fractional Laplace motion shows a multifractal behavior in the scales under consideration. (c) Change of PDF of increments of simulated fractional Laplace motion series. Solid dots correspond to PDF of increments at $r=10$ and + to $r=60$. Solid line indicates a Gaussian PDF.

scales in log-log space for an arbitrary choice of the parameter values, $H=0.4$ and $\nu=3.0$. (These values of H and ν are used for illustration of the model properties and the estimation of these parameters is discussed more thoroughly in the next subsection). It is to note that from Fig. 4(a) that although Eq. (12) does not analytically accept a power-law expression on r, for all practical purposes, fractional Laplace motion can be approximated by a self-similar process, i.e., the structure functions show a power-law relationship in scales at least for the range of scales which coincide with the scaling regime of sediment transport series (scales or lags of $r=4$ to $r=64$). Plotting the $\tau(q)$ curve [estimated from the slopes of $M(q,r)$ vs r in log-log space within the above scaling regime] one can see that the scaling exponents, $\tau(q)$, show a nonlinear dependence on the order of moments q [see Fig. 4(b)]. It is to note that the scaling exponents $\tau(q)$ are independent of the variance of the parent process σ^2. The change in PDF of the increments of fractional Laplace motion with scale is shown in Fig. 4(c), where the PDF at small scales [$r=10$ in Fig. 4(a)] shows a double-exponential behavior and it eventually tends to a Gaussian distribution for large scales [$r=60$ in Fig. 4(a)]. The above results document that fractional Laplace motion can be approximated by a stochastic self-similar process in an intermediate range of scales and within those scales it exhibits a multifractal behavior. At the limit of very large time scales, i.e., as $r\to\infty$, fractional Laplace motion tends to a fractional Brownian motion with $\tau(q)$ a linear function of q (i.e., monofractal behavior).

It is interesting to note from Eq. (12) that the second-order structure function of Laplace motion ($H=0.5$ and $q=2$) obeys a power-law relationship in scales and in particular it shows a linear dependence on scales

$$M(2,r) = \left[\frac{2}{\nu\sqrt{\pi}}\Gamma(1.5) \right] r, \tag{13}$$

yielding an exponent of $\tau(2)=1$. This implies that Laplace motion has self-similar second-order moments, i.e., it shows a log-log linear power spectrum (although higher order moments are not exact power laws). In the next subsection we elaborate on the parameter estimation of the fractional Laplace motion from the sediment transport series.

B. Model fitting

As seen in the previous section, fractional Laplace motion has three parameters H, ν and σ. The scale parameter of the operational time PDF, β, is 1 by the definition of fractional Laplace motion [27]. Estimation of the parameters H and ν from the sediment transport series is performed by minimizing the mean squared error between the empirical and theoretical $\tau(q)$ curves. The mean squared error, denoted by MSE, is a function of H and ν and is independent of σ,

$$\mathrm{MSE}(H,\nu) = \sum_q [\tau_m(q) - \hat{\tau}(q)]^2, \tag{14}$$

where $\hat{\tau}(q)$ are the estimated scaling exponents of the sediment transport series [see Fig. 3(b)] and $\tau_m(q)$ are the scaling exponents of the fractional Laplace motion model which are computed from the slopes of the theoretical $M(q,r)$ versus r within the scaling regime of the sediment transport series ($4<r<64$) in the log-log space [see Fig. 4(b)]. Minimization of the mean squared error for the sediment transport series yields a Hurst exponent of $H=0.39$ and a shape parameter of $\nu=6.8$. It is to note that the multiscale structure of fractional Laplace motion model is determined by the parameters H and ν. Further, we estimate the parameter σ by minimizing the mean squared error between the variance of the increments of sediment transport series and the fractional Laplace noise for $H=0.39$ and $\nu=6.8$ over the scaling regime ($4<r<64$),

$$\sigma = \mathrm{Min} \sum_{r=4}^{r=64} \{\mathrm{Var}[\delta S(t,r)] - \mathrm{Var}[Y(t,r)]\}^2, \tag{15}$$

where $\mathrm{Var}[\delta S(t,r)]$ is the variance of the increments of sediment transport series and $\mathrm{Var}[Y(t,r)]$ is the variance of fractional Laplace noise at the scale r, given by Eq. (10). The value of σ estimated using Eq. (15) was $\sigma=0.296$. The multifractal parameters of the fractional Laplace motion model computed with the estimated parameters of $H=0.39$ and $\nu=6.8$ were $c_1=0.41$ and $c_2=0.041$, which compare very well to the values estimated from the sediment transport data of $c_1=0.41$ and $c_2=0.04$. [Note that c_1 and c_2 were not used directly in the model fitting which was done via Eq. (14) on the whole $\tau(q)$ curve]. As a result the model and the data-estimated $\tau(q)$ curves are indistinguishable. Figure 5(a) shows the increments of sediment transport series at a scale of $r=20$ or 40 min (note that this scale lies within the scaling regime of the sediment transport series). For visual comparison, the fractional Laplace noise simulated series with the estimated parameters $H=0.39$, $\nu=6.8$, and $\sigma=0.296$ at the same scale is shown in Fig. 5(b).

As noted in the previous section, fractional Laplace noise is negatively correlated for $H<0.5$. Figure 6(a) shows the autocorrelation function of the increments of sediment transport series at the scale $r=20$ (40 min). The data show a negative correlation in the scaling regime of the sediment transport series for small lags. This is qualitatively consistent with the fractional Laplace noise model which shows a negative correlation for the estimated parameter values [see Fig. 6(b)]. The increments of fractional Laplace motion at small scales follow a Laplace PDF which eventually becomes Gaussian at larger scales. Figure 7(a) shows the PDF of sediment transport increments at a scale of $r=4$ which is the beginning of the scaling regime of the sediment transport series. A Laplace PDF provides a good fit to the increments at that scale. As noted in Fig. 7(b), the PDF of sediment transport increments at a scale of $r=64$ (128 min) tends to a Gaussian PDF. Thus, one can see that the sediment transport series are consistent with the properties of fractional Laplace motion within the scaling regime.

V. DISCUSSION

In the previous section we established the fact that the fractional Laplace motion model is able to reproduce the intricate stochastic structure of the observed sediment transport series over a range of scales and also reproduce the

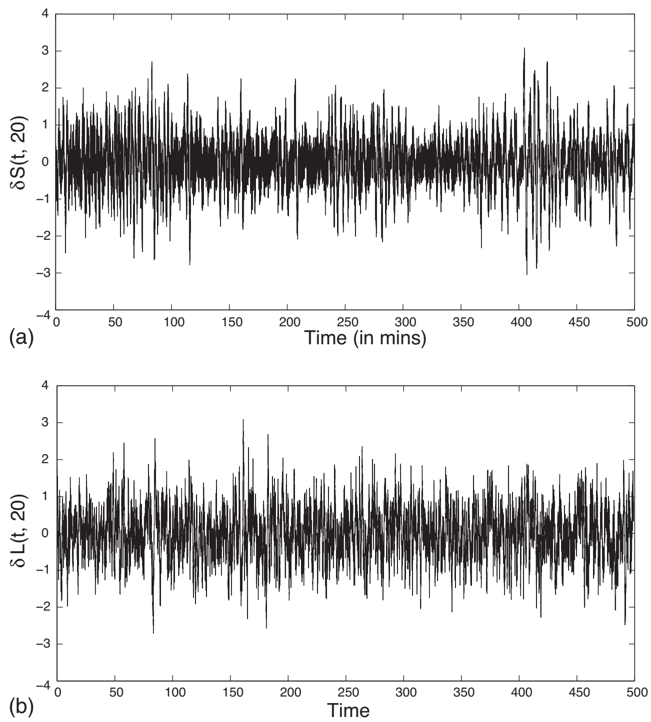

FIG. 5. (a) Comparison of the increments of the sediment transport series in kgs at scale $r=20$ (40 min) and (b) the same scale increments of simulated fractional Laplace motion series with $H=0.39$, $\nu=6.8$, and $\sigma=0.3$. The values of H and ν were obtained by minimizing the mean squared error defined in Eq. (14). The value of σ was obtained using Eq. (15). The scale of $r=20$ was chosen for comparison as it lies within the scaling regime of the sediment transport series.

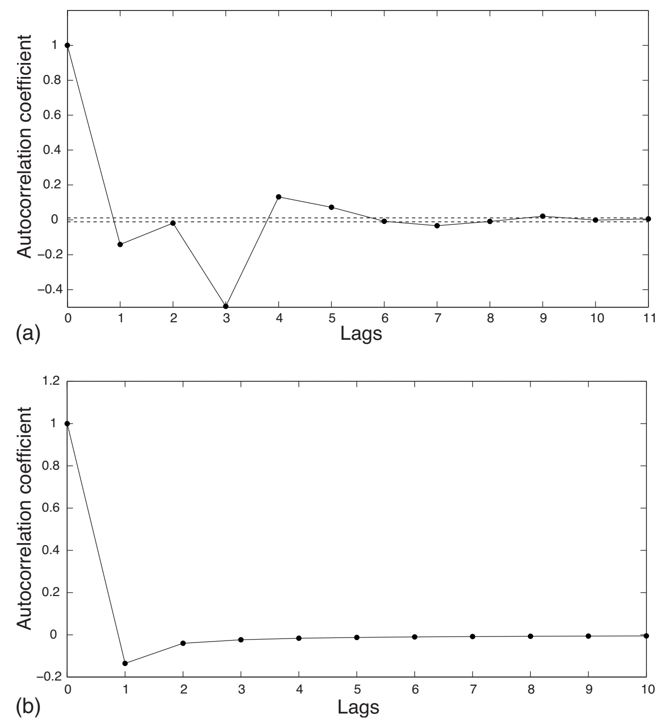

FIG. 6. (a) The autocorrelation function of the increments of sediment transport series at a scale of $r=20$ (40 min). The dashed lines indicate the 95% confidence intervals (approximated as $\pm 1.96/\sqrt{N}$, $N=30\,293$ points) on the autocorrelation coefficients. (b) The autocorrelation function of generated fractional Laplace noise series at the same scale with parameters $H=0.39$ and $\nu=6.8$ fitted to the data. The autocorrelation of the fractional Laplace noise is computed from Eqs. (10) and (11).

change of the PDFs of increments of sediment transport series in the scaling regime. In this section, we discuss the physical significance of the notion of operational time in sediment transport series. Near-bed turbulence is known to play an important role in sediment transport [12]. Turbulent velocity fluctuations pick up sediment particles and transport them over long distances. However, since the turbulent velocities themselves are known to exhibit variability over a large range of scales, the entrainment time experienced by the sediment particles is also expected to carry some of this variability. This consideration leads to a randomization of time over which a sediment particle is operated upon, as sediment particles in different velocity zones experience time to move faster or slower depending on whether they are in a high- or low-velocity zone, respectively. Thus, the notion of operational time can arise due to the stochastic nature of sediment particle entrainment. It is interesting to note that the turbulent velocity fluctuations themselves exhibit Laplace and stretched Laplace distributions at small scales and their PDFs become Gaussian at larger scales [33]. It is also interesting to note that the rate of sediment particle entrainments, which are proportional to the shear stress fluctuations at the bed, have been reported to follow a Gamma distribution [9]. Both these observations are qualitatively consistent with the fractional Laplace motion model for sediment transport proposed in this paper.

The observed multiscaling and intermittency in sediment transport series (macroscale behavior) was shown to arise by the introduction of the notion of operational time in Brownian-type particle movement (microscale behavior). Thus, while the model parameters H and ν relate to the (unobserved) particle movement statistics, they are estimated from the (observed) sediment transport statistics, and specifically from their multiscale behavior concisely parameterized via the parameters c_1 and c_2. It is of interest to study how the parameter space of (H,ν) relates to that of (c_1,c_2) in order to gain insight on model sensitivity and the physical meaning of the parameter ν which characterizes the variability of the particle motion. We compute the multifractal parameters c_1 and c_2 for different values of the model parameters H and ν by evaluating $M(q,r)$ from Eq. (12), estimating $\tau(q)$ in the range $4 \leq r \leq 64$, and approximating the $\tau(q)$ curve as a quadratic function in q [Eq. (4)]. Figure 8 shows the contour plots of c_1 and c_2 for different values of H and ν. It is to note that the average "roughness" of the sediment series, quantified by the parameter c_1, is strongly dependent on the Hurst exponent of the fractional Brownian motion H [see Fig. 8(a)] and not as much on the parameter ν of the operational time. On the other hand, from Fig. 8(b), one can see that the intermittency coefficient c_2 is strongly dependent on the shape parameter ν of the distribution of operational time for a given value of H. In particular, for a given value of H, the value of c_2 is higher for a higher value of ν. One way to understand this is to note that for higher values of ν the Gamma distribution has a higher variance. Thus, for higher

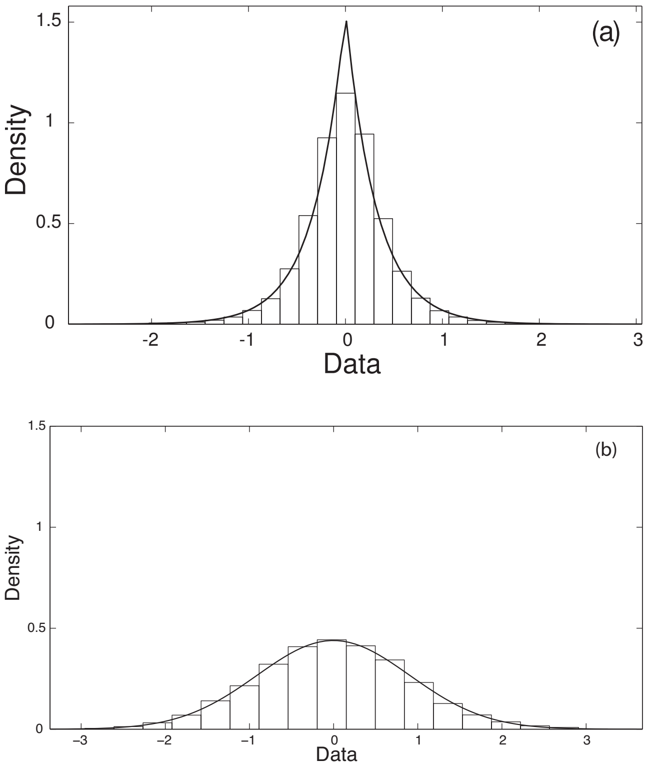

FIG. 7. Change in PDF of sediment transport increments in the scaling regime. (a) Laplace PDF (solid line) provides a good fit to the PDF of sediment transport increments at $r=4$ (8 min; beginning of the scaling regime) and (b) the PDF of sediment transport increments becomes Gaussian (solid line) at $r=64$ (128 min; ending of the scaling regime).

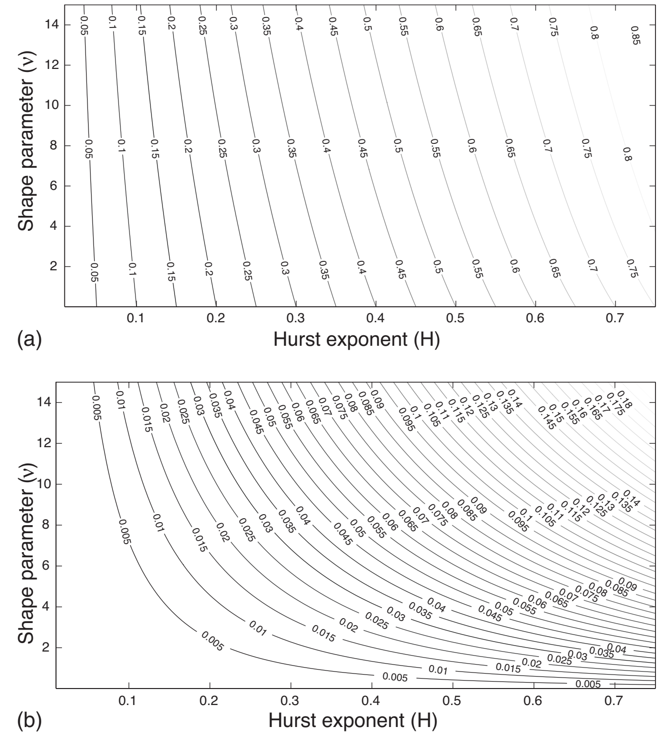

FIG. 8. Contour plots of the multifractal parameters, (a) c_1 and (b) c_2, for different values of the fractional Laplace motion model parameters H and ν.

values of ν the operational time is sampled from a distribution with higher variance and this variability in the operational time shows up as a higher intermittency coefficient in the sediment transport series (larger degree of temporal heterogeneity in bursts of sediment transport increments). It is emphasized that estimation of the parameter values of the fractional Laplace motion, H and ν, was performed through the scaling exponents of the structure functions of the sediment transport series [Eq. (14)]. Direct estimation of the parameters H and ν, or for that matter direct assessment of the whole statistical structure of operational time from observations, would require access to series of particle entrainment which are difficult to make and are not available in the experimental setting studied here. Rather, the present study attempted a physical insight via relating the macroscale statistics of the sediment series to the microscale dynamics of particle movement.

VI. CONCLUDING REMARKS

In this work we proposed the adaptation of fractional Laplace motion as a stochastic model for sediment transport. Fractional Laplace motion arises from randomization of the clock time in fractional Brownian motion, and introduces the notion of operational time. The physical significance of operational time in the context of sediment transport was reasoned on the basis that the stochastic nature of turbulent velocity fluctuations near the bed induces stochasticity in particle entrainment and, therefore, the time over which particles are in motion. The proposed model was shown able to reproduce the multiscale statistics of sediment transport series and was validated against a data set from a large-scale laboratory experiment. The effect of the model parameters on the multifractal parameters of sediment transport series was also discussed. Although direct estimation of the model parameters would require particle-scale observations, it was shown here that an indirect estimation based on the statistics of sediment transport series is possible. We see this work as a step toward relating the microscale dynamics of particle movement to the macroscale statistics of sediment transport via minimum complexity stochastic models.

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