Testing for Multifractality and Multiplicativity using Surrogates

E. Foufoula-Georgiou (Univ. of Minnesota)
S. Roux & A. Arneodo (Ecole Normale Superieure de Lyon)
V. Venugopal (Indian Institute of Science)

Contact: efi@umn.edu

AGU meeting, Dec 2005
Motivating Questions

- Multifractality has been reported in several hydrologic variables (rainfall, streamflow, soil moisture etc.)

- Questions of interest:
  - What is the nature of the underlying dynamics?
  - What is the simplest model consistent with the observed data?
  - What can be inferred about the underlying mechanism giving rise to the observed series?
Precipitation: Linear or nonlinear dynamics?

- Multiplicative cascades (MCs) have been assumed for rainfall motivated by a turbulence analogy (e.g., Lovejoy and Schertzer, 1991 and others)

- Recently, Ferraris et al. (2003) have attempted a rigorous hypothesis testing. They concluded that:
  - MCs are not necessary to generate the scaling behavior found in rain
  - The multifractal behavior of rain can be originated by a nonlinear transformation of a linearly correlated stochastic process.
Methodology

- Test null hypothesis:
  - \( H_0 \): Observed multifractality is generated by a linear process
  - \( H_1 \): Observed multifractality is rooted in nonlinear dynamics

- Compare observed rainfall series to “surrogates”

- Surrogates destroy the nonlinear dynamical correlations by phase randomization, but preserve all other properties (Thieler et al., 1992)
Purpose of this work

- Introduce more discriminatory metrics which can depict the difference between processes with non-linear versus linear dynamics.

- Illustrate methodology on generated sequences (FIC and RWC) and establish that “surrogates” of a pure multiplicative cascade lack long-range dependence and are monofractals.

- Test high-resolution temporal rainfall and make inferences about possible underlying mechanism.
Metrics

1. **WTMM Partition function:** \( q = 1, 2, 3 \ldots \)

\[
Z(q,a) = \sum_{l(a)} |T_a(x)|^q
\]
\( \mathbb{L}(a) \) – set of maxima lines at scale \( a \)

2. **Cumulants** \( C_n(a) \) **vs.** \( a \)

\[
C_1(a) \equiv \langle \ln \ | T_a \rangle \sim c_1 \ln(a)
\]
\[
C_2(a) \equiv \langle \ln^2 \ | T_a \rangle - \langle \ln \ | T_a \rangle^2 \sim c_2 \ln(a)
\]
\[
C_3(a) \equiv \langle \ln^3 \ | T_a \rangle - 3\langle \ln^2 \ | T_a \rangle\langle \ln \ | T_a \rangle + \langle \ln \ | T_a \rangle^3 \sim c_3 \ln(a)
\]

etc.

Recall \( \tau(q) = -c_0 + c_1 q - c_2 \frac{q^2}{2} + \cdots \)

\[D(h) = \min_q (qh - \tau(q))\]

3. **Two-point magnitude correlation analysis**

\[
C(a, \Delta x) = \left\langle \left( |T_a(x)| - \ln |T_a(x)| \right) \left( |T_a(x+\Delta x)| - \ln |T_a(x+\Delta x)| \right) \right\rangle
\]

\[
C(a, \Delta x) \sim \ln \Delta x, \quad \Delta x > a \implies \text{long-range dependence}
\]

\[
C(a, \Delta x) \sim -c_2 \ln \Delta x \quad \implies \text{multiplicative cascade}
\]

\[
(C_2(a) \sim -c_2 \ln a)
\]
Surrogate of an FIC

a) FIC: $c_1 = 0.13; \quad c_2 = 0.26; \quad H^* = 0.51$

(To imitate rain: $c_1 = 0.64; \quad c_2 = 0.26$)

b) Surrogates

![FIC](image)

![Surrogate](image)
Multifractal analysis of FIC and surrogates
(Ensemble results)

\[ \ln[Z(q,a)] \]

\[ \ln(a) \]

Cannot distinguish FIC from surrogates

\( q = 1 \)

\( q = 2 \)

\( q = 3 \)

\( o \rightarrow \text{Avg. of 100 FICs} \)

\( * \rightarrow 100 \text{ Surrogates of 100 FICs} \)
Cumulant analysis of FIC and surrogates

(Ensemble results)

\[ C(n,a) \]

\[ \ln(a) \]

\[ n = 1 \]
\[ n = 2 \]
\[ n = 3 \]

Easy to distinguish FIC from surrogates

\( o \rightarrow \) Avg. of 100 FICs
\( * \rightarrow \) 100 Surrogates of 100 FICs
Bias in estimate of $c_1$ in surrogates

$$\tau(q) = -c_0 + c_1 q - c_2 \frac{q^2}{2} + \cdots$$

$$\tau(2) = -c_0 + 2c_1 - 2c_2 + \cdots$$

$\tau(2)$ is preserved in the surrogates

FIC ($c_1 = 0.64$; $c_2 = 0.26$) $\rightarrow$ Surrogates ($c'_1 = 0.38$; $c'_2 \approx 0$)
Effect of sample size on $c_1$, $c_2$ estimates
(FIC vs. Surrogates)

True FIC
($c_1 = 0.64$)

Surrogates
($c_1' = 0.38$)

Surrogates
($c_2' \cong 0$)

True FIC

$\circ \rightarrow$ FIC
$\star \rightarrow$ Surrogates
Two-point magnitude analysis

![Graph showing Two-point magnitude analysis results with FIC and Surrogate labels.]
Rainfall vs. Surrogates

(a) Rainfall

(b) Surrogate
Multifractal analysis of Rain and surrogates

\[ \ln \left[ Z(q,a) \right] \quad \ln (a) \]

\( q = 1 \) (b)
\( q = 2 \) (c)
\( q = 3 \) (d)

Hard to distinguish Rain from surrogates

\( \circ \rightarrow \text{Rain} \)
\( * \rightarrow \text{Surrogate} \)
Cumulant analysis of Rain and surrogates

(a) $n = 1$

(b) $n = 2$

(c) $n = 3$

$C(n,a)$

$\ln(a)$

Easy to distinguish Rain from surrogates

$\circ \rightarrow$ Rain
$\ast \rightarrow$ Surrogate
Two-point magnitude analysis
Rain vs. Surrogates

\[ C_{aa}(\Delta t) \]

\[ \ln(\Delta t) \]
Conclusions

- Surrogates can form a powerful tool to test the presence of multifractality and multiplicativity in a geophysical series
  - Using proper metrics (wavelet-based magnitude correlation analysis) it is easy to distinguish between a pure multiplicative cascade (NL dynamics) and its surrogates (linear dynamics)
  - The simple partition function metrics have low discriminatory power and can result in misleading interpretations
- Temporal rainfall fluctuations exhibit NL dynamical correlations which are consistent with that of a multiplicative cascade and **cannot** be generated by a NL filter applied on a linear process
- The use of fractionally integrated cascades for modeling multiplicative processes needs to be examined more carefully (e.g., turbulence)
An interesting result...

- Surr(FIC): Observed Linear $\tau(q)$ for $q < 2$ and NL for $q > 2$
- Suggests a “Phase Transition” at $q \approx 2$
- $\tau(q)$ from cumulants captures behavior at around $q = 0$ (monofractal)
- Suspect FI operation: preserves multifractality but not the multiplicative dynamics → Test a pure multiplicative cascade (RWC)
An interesting result …

FIC vs. Surrogates

RWC vs. Surrogates

IS "Fractionally Integrated Cascade" A GOOD MODEL FOR TURBULENCE OR RAINFALL?
END
Conclusions on Surrogates

- The surrogates of a multifractal/multiplicative function destroy the long-range correlations due to phase randomization.

- The surrogates of an FIC show a “phase transition” at around $q=2$ ($q<2$ monofractal, $q>2$ multifractal). This is because the strongest singularities are not removed by phase randomization.

- The surrogates of a pure multiplicative multifractal process (RWC) show monofractality.

- Recall that FIC results from a fractional integration of a multifractal measure and thus itself is not a pure multiplicative process.

- Implications of above for modeling turbulence with FIC remain to be studied (surrogates of turbulence show monofractality but surrogates of FIC do not).
Bias in estimate of $c_1$ in surrogates

$$\tau(q) = -c_0 + c_1 q - c_2 \frac{q^2}{2} + \cdots$$

$$\tau(2) = -c_0 + 2c_1 - 2c_2$$

$$\tau(3) = -c_0 + 3c_1 - \frac{9c_2}{2}$$

FIC: $c_1 = 0.64; c_2 = 0.26$

$$\tau(2) = -c_0 + 2c_1 - 2c_2 = -1 + 2(0.64 - 0.26) = -0.24$$

$$\tau(3) = -c_0 + 3c_1 - \frac{9c_2}{2} = -1 + 3(0.64) - \frac{9(0.26)}{2} = -0.25$$

Surrogates: $c_1', c_2'$

$\tau(2)$ is preserved; $c_2' = 0$

$C_1' = 0.38$

$$\tau(2) = -c_0' + 2(c_1' - c_2') \Rightarrow c_1' = \frac{(\tau(2) + c_0')}{2} + c_2'$$

$$\Rightarrow c_1' = 0.38$$

$$\tau(3) = -c_0' + 3c_1' - \frac{9c_2'}{2} = -1 + 3(0.38) - \frac{9(0)}{2} = 0.14$$
Multifractal Spectra: $\tau(q)$ and $D(h)$
(FIC vs. Surrogates)

$c_1 = 0.64; c_2 = 0.26$
3 slides - RWC vs. Surrogates

c_1 = 0.64; c_2 = 0.26
Multifractal analysis of RWC and surrogates
(Ensemble results)

$n = 1$

$n = 2$

$n = 3$

$\ln (a)$

$\ln [ Z(q,a) ]$

$\ln (a)$

$\text{Cannot distinguish RWC from surrogates}$

$c_1 = 0.64; c_2 = 0.26$

RWC - Random Wavelet Cascade

$\circ \rightarrow \text{Avg. of 100 RWC}$

$\ast \rightarrow \text{100 Surrogates of 100 RWCs}$
Cumulant analysis of RWC and surrogates
(Ensemble results)

\[ C(n,a) \]

Easy to distinguish RWC from surrogates

\[ n = 1 \]

\[ n = 2 \]

\[ c_1 = 0.64; c_2 = 0.26 \]
Multifractal Spectra: $\tau(q)$ and $D(h)$

(RWC vs. Surrogates)

$c_1 = 0.64; c_2 = 0.26$
3 slides - FIC vs. Surrogates

\[ c_1 = 0.64; \ c_2 = 0.10 \]
Cumulant analysis of FIC and surrogates
(Ensemble results)

\[ C(n,a) \]

\[ \ln(a) \]

\[ n = 1 \]

\[ n = 2 \]

\[ n = 3 \]

Easy to distinguish FIC from surrogates

\( c_1 = 0.64; \ c_2 = 0.10 \)

\( o \rightarrow \text{Avg. of 100 FICs} \)

\( * \rightarrow 100 \text{ Surrogates of 100 FICs} \)
**Multifractal Spectra: \( \tau(q) \) and \( D(h) \)**

(FIC vs. Surrogates)

\[
\begin{align*}
\tau(q) & = 0.64; \\
D(h) & = 0.10
\end{align*}
\]
Multifractal analysis of FIC and surrogates
(Ensemble results)

\[ \ln [ Z(q,a) ] \]

\[ \ln (a) \]

Cannot distinguish FIC from surrogates

- \( q = 1 \)
- \( q = 2 \)
- \( q = 3 \)

- \( \circ \rightarrow \text{Avg. of 100 FICs} \)
- \( * \rightarrow \text{100 Surrogates of 100 FICs} \)