Revisiting Multifractality of High Resolution Temporal Rainfall:

New Insights from a Wavelet-Based Cumulant Analysis

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Motivating Questions

- Is scale invariance present in rainfall? Over what scales? What type of scale invariance?
- Does the nature of scaling vary considerably from storm to storm or is it universal? Does it relate to any physical parameters?
- What models are consistent with the scaling structure of rainfall observations and what inferences can be made about the underlying generating mechanism?
- Practical use of scale invariance for sampling design, downscaling and prediction of extremes?

3 talks on Rainfall

- Introduction of powerful diagnostic methodologies for multifractal analysis: new insights for high resolution temporal rainfall

  (H31J-06: This one)

- Analysis of simultaneous series of rainfall, temperature, pressure, and wind in an effort to relate statistical properties of rain to those of the storm meteorological environment

  (H31J-07 - next talk: Air Pressure, Temperature and Rainfall: Insights From a Joint Multifractal Analysis)

- Methodologies for discriminating between linear vs. nonlinear dynamics and implications for rainfall modeling

  (H32C-07 @ 11:50AM: Testing multifractality and multiplicativity using surrogates)

Please visit our posters…

(H33E: Wedn., 1:40PM, MCC Level 2- posters)

Geomorphology:

- River corridor geometry: Scaling relationships and confluence controls
  C. Gangodagamage, E. Foufoula-Georgiou, W. E. Dietrich

- Scale Dependence and Subgrid-Scale Closures in Numerical Simulations of Landscape Evolution
  P. Passalacqua, F. Porté-Agel, E. Foufoula-Georgiou, C. Paola

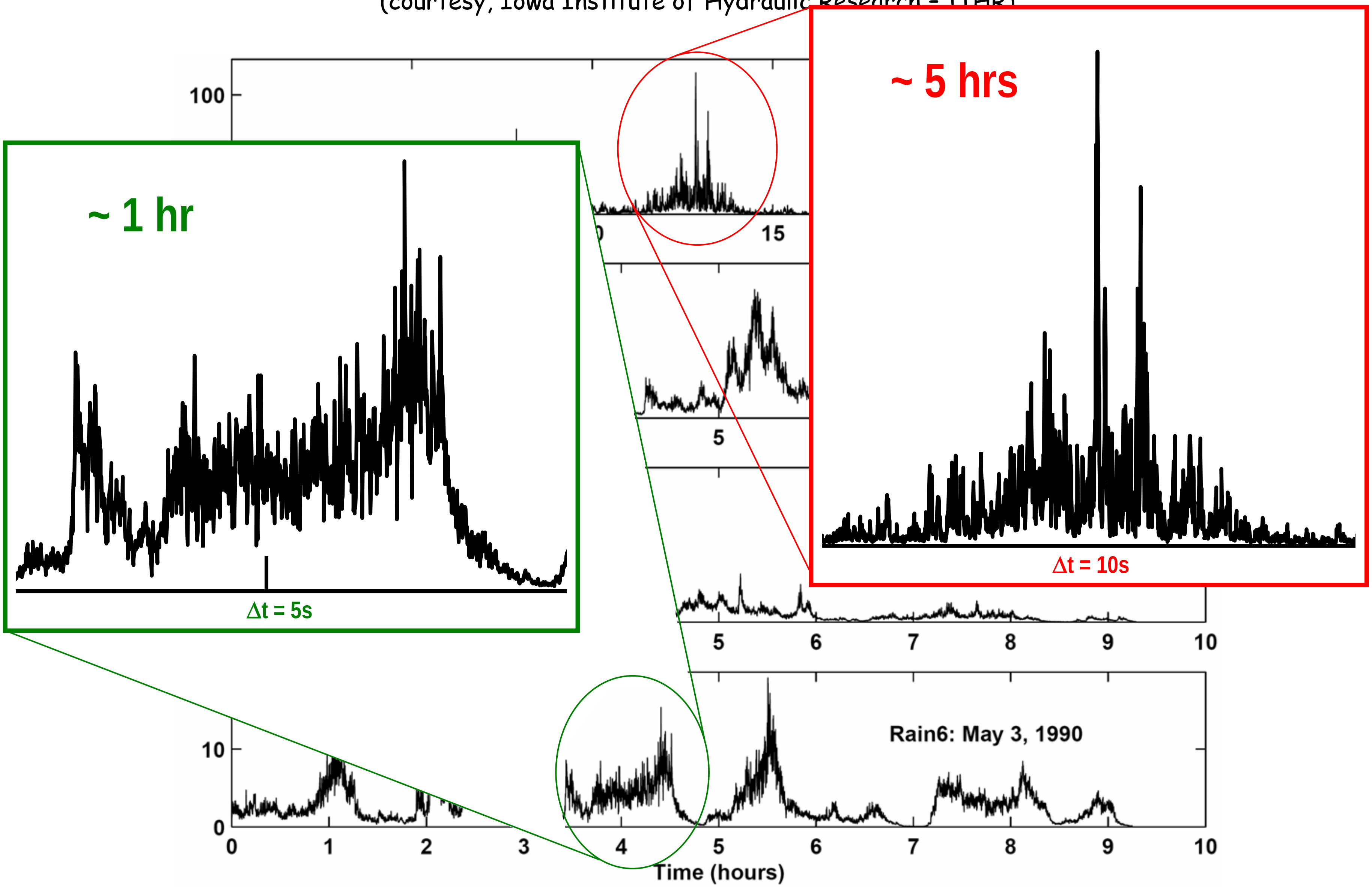
Hydrologic response and floods:

- Scaling in Hydrologic Response and a Theoretical Basis for Derivation of Probabilistic Synthetic Unit Hydrographs
  R. K. Shrestha, I. Zaliapin, B. Dodov, E. Foufoula-Georgiou

- Floods as a mixed-physics phenomenon: Statistics and Scaling
  N. Theodoratos, E. Foufoula-Georgiou

High-resolution temporal rainfall data

(courtesy, Iowa Institute of Hydraulic Research - IIHR)



Multifractal Formalism

(Parisi and Frisch, 1985)

- $< | u(x+l) - u(x) |^q > \sim l^{\tau(q)}$ — $\tau(q)$ is NL function of q

- NL behavior of $\tau(q)$ was interpreted as existence of a heterogeneity in the local regularity of the velocity field

- $| u(x_0) - u(x_0 - \varepsilon) | \sim \varepsilon^{h(x_0)}$ — $h(x_0)$ = Hölder exponent

  $0 \leq h \leq 1$

  h = 0 discontinuity in function

  h = 1 discontinuity in derivative

- $D(h) = d_H[X, h(x) = h]$ — d_H = Hausdorff dimension

- $D(h) = \min_q [qh - \tau(q)]$ — Legendre transform

Limitations of structure function method

- If nonstationarities not removed by 1st order differencing, bias in inferences and scaling exponent estimates

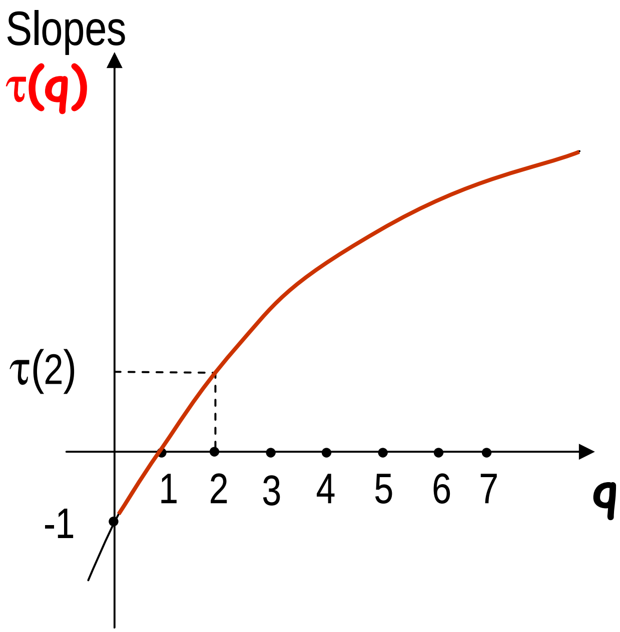
- The largest singularity that can be identified is $h_{max} = 1$

- Need to go to high order moments to reliably estimate a nonlinear $\tau(q)$ curve

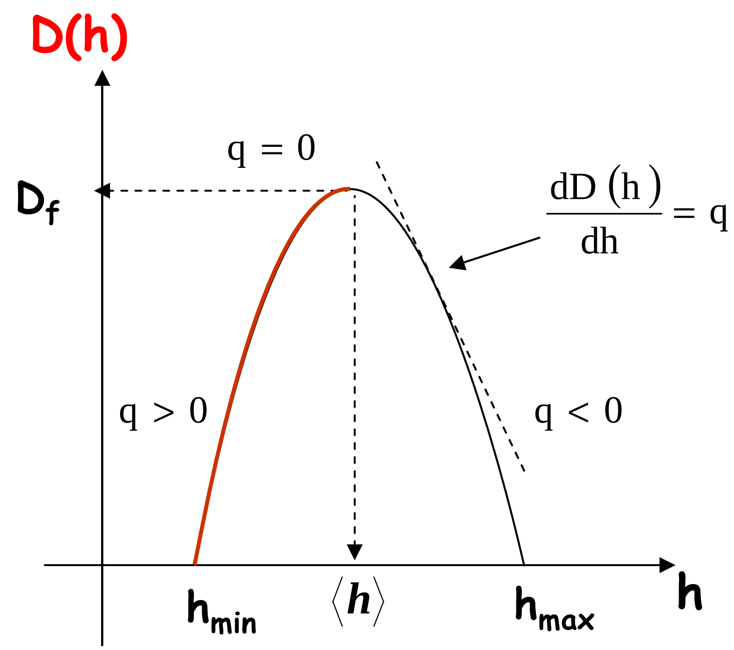
- PDF of 1st order increments is centered at zero. Cannot take negative moments ($q < 0$) as might have divergencies $\rightarrow$ Have access only to the increasing part of D(h) [for $q > 0$]

Multifractal Spectra

Spectrum of scaling exponents



Spectrum of singularities



Wavelet-based multifractal formalism

(Muzy et al., 1993; Arneodo et al., 1995)

- CWT of *f(x)* : $T_{\psi}[f](x,a) = \frac{1}{|a|} \int f(u)\ \psi\left(\frac{u-x}{a}\right) du$

- The local singularity of *f(x)* at point x_0 can be characterized by the behavior of the wavelet coefficients as they change with scale, provided that the order of the analyzing wavelet $n > h(x_0)$

- Can obtain robust estimates of $h(x_0)$ using "maxima lines" only: $T_a(x)$ i.e. WTMM

$$\left| T_a(x_0) \right| \sim a^{h(x_0)} \qquad a \to 0$$

- It can be shown that

$$\left\langle \left| T_a(x_0) \right|^q \right\rangle \sim a^{\tau(q)+D_f}$$

$f(x)$

Structure Function
Moments of
$|f(x+l) - f(x)|$

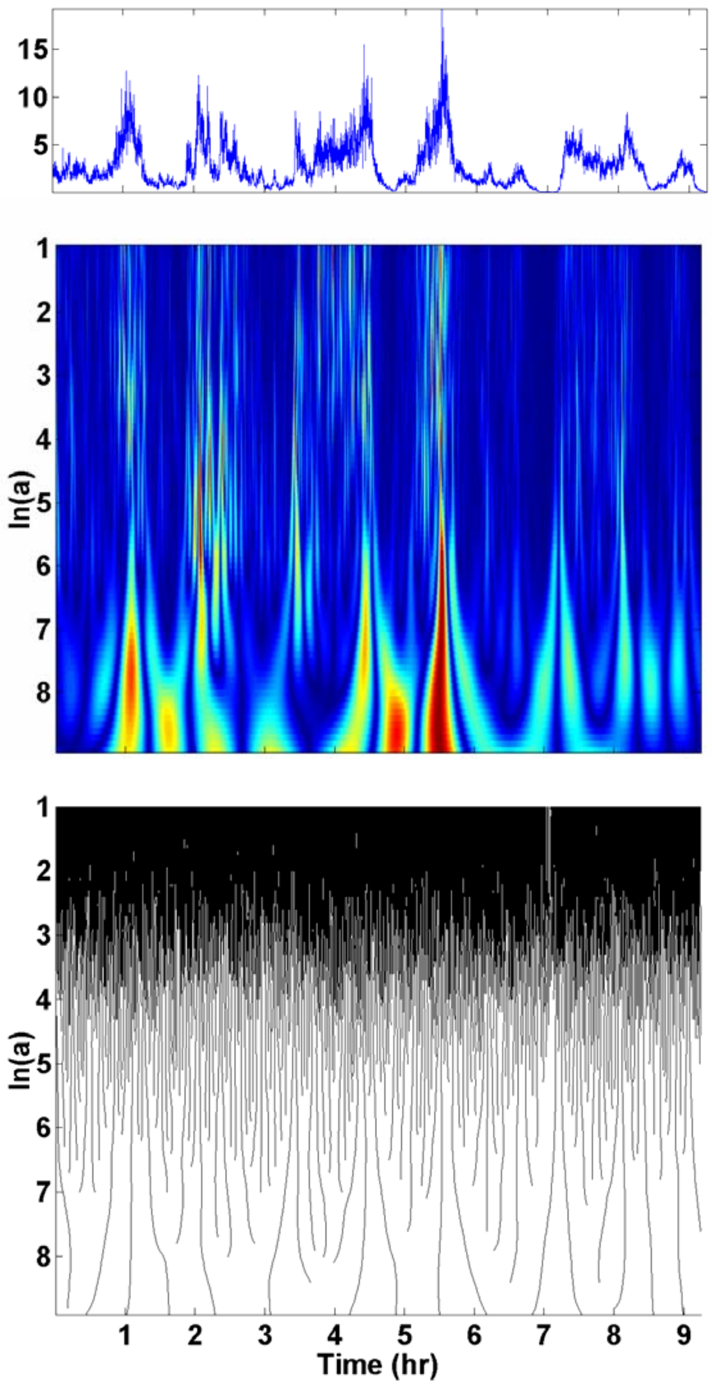
$T_\Psi[f](x,a)$

Partition Function
Moments of
$|T_\Psi[f](x,a)|$

WTMM
$T_a(x)$

→ **Partition Function**
Moments of $|T_a(x)|$
(access to $q < 0$)

→ **Cumulant analysis**
Moments of $\ln |T_a(x)|$
(direct access to statistics of singularities)



Magnitude Cumulant Analysis

- Estimate τ(q) and D(h) without the need to compute higher order moments of the data

- Start with $\ln\left\langle \left|T_a(x)\right|^q \right\rangle \sim \tau(q) + D_f$ ①

- Form the cumulant generating function = log (characteristic function)

$$\Psi_a(q) = \ln\left\langle e^{q \ln\left|T_a(x)\right|} \right\rangle = \sum_{u=1}^{\infty} C_n(a) \frac{q^n}{n!}$$ ②

- From ① and ② $$\boldsymbol{\tau(q) = -c_0 + c_1 q - c_2 \frac{q^2}{2} + \ldots}$$

where

$$c_0 = D_f, \quad c_1 = C_1(a)/\ln a, \quad c_2 = -C_2(a)/\ln a$$

$$C_1(a) = \left\langle \ln\left|T_a\right| \right\rangle, \quad C_2(a) = \left\langle \ln\left|T_a\right| - \overline{\ln\left|T_a\right|} \right\rangle^2$$ etc.

Magnitude Cumulant Analysis

- Compute cumulants of $\ln|T_a|$

$$C_1(a) \equiv \langle \ln |T_a| \rangle$$

$$C_2(a) \equiv \langle \ln^2 |T_a| \rangle - \langle \ln |T_a| \rangle^2$$

$$C_3(a) \equiv \langle \ln^3 |T_a| \rangle - 3\langle \ln^2 |T_a| \rangle \langle \ln |T_a| \rangle + \langle \ln |T_a| \rangle^3$$

etc.

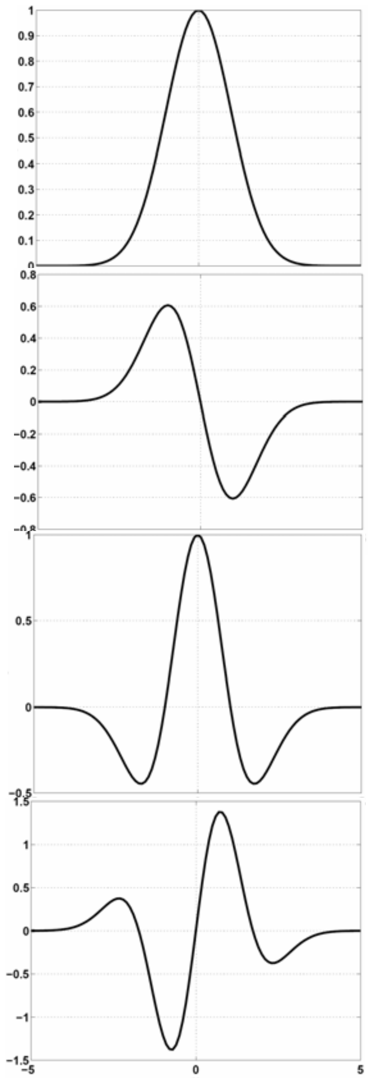
- For a multifractal:

  - ✓ $C_1(a) \sim c_1 \ln(a)$; $\quad C_2(a) \sim -c_2 \ln(a)$; $\quad C_3(a) \sim c_3 \ln(a)$

  - ✓ $\tau(q) = -c_0 + c_1 q - c_2 \frac{q^2}{2} + \cdots$ $\qquad c_2 \neq 0$ ➔ multifractal

  - ✓ c_n directly relate to the statistical moments of the singularities **h(x)**

$$\langle h \rangle = c_1 \,; \quad \mathrm{var}(h) = \frac{-c_2}{\ln a} \,; \quad h_{\min,\max} = c_1 \mp \sqrt{2 c_2 c_0} \quad \text{for a LN multifractal}$$

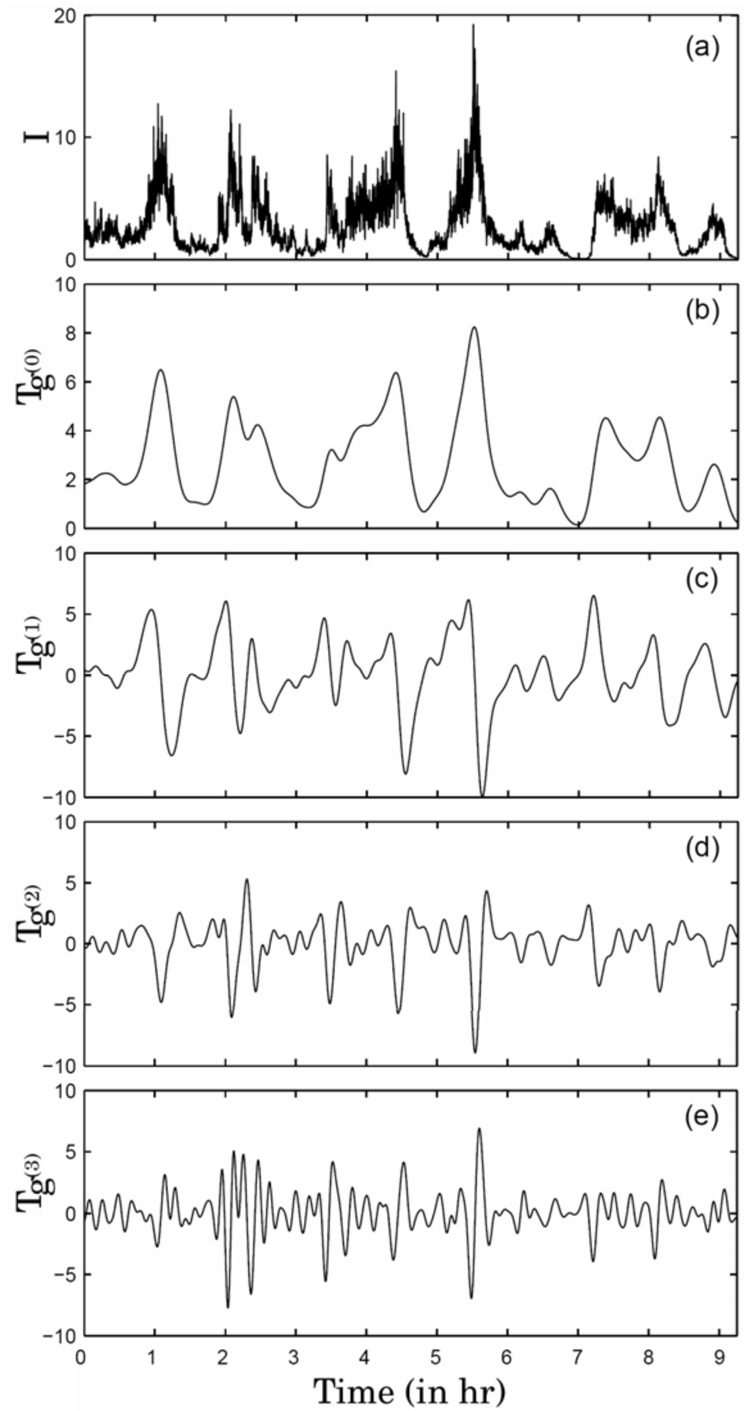


$g^{(0)}$

$g^{(1)}$

$g^{(2)}$

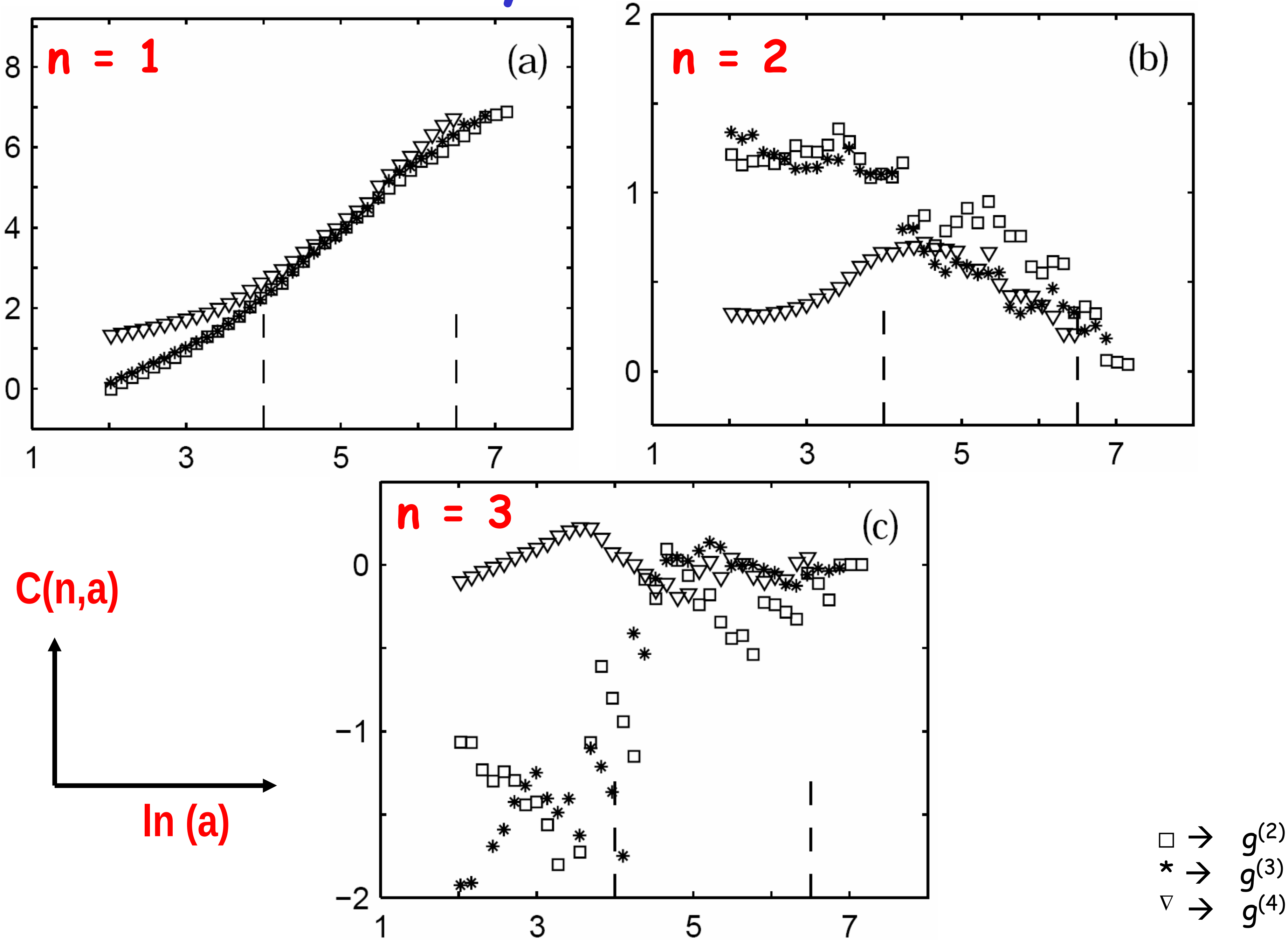
$g^{(3)}$



Rain6

Rainfall fluctuations
(at scale = 740s ≅ 12 min)

Cumulant Analysis of Rain6 Cumulative



Cumulant Estimates of Rain 6 Intensity with $g^{(n)}$, n = 0,1,2,3

	c_0^I	c_1^I	c_2^I	c_3^I
$g^{(0)}$	0.94 ± 0.05	0.11 ± 0.02	0.15 ± 0.02	~ 0
$g^{(1)}$	0.95 ± 0.04	0.54 ± 0.03	0.28 ± 0.05	~ 0
$g^{(2)}$	0.98 ± 0.02	0.64 ± 0.03	0.26 ± 0.04	~ 0
$g^{(3)}$	1.00 ± 0.02	0.69 ± 0.06	0.24 ± 0.05	~ 0

$$c_0^I = c_0^c; \quad c_1^I = c_1^c - 1; \quad c_2^I = c_0^c; \quad \ldots$$

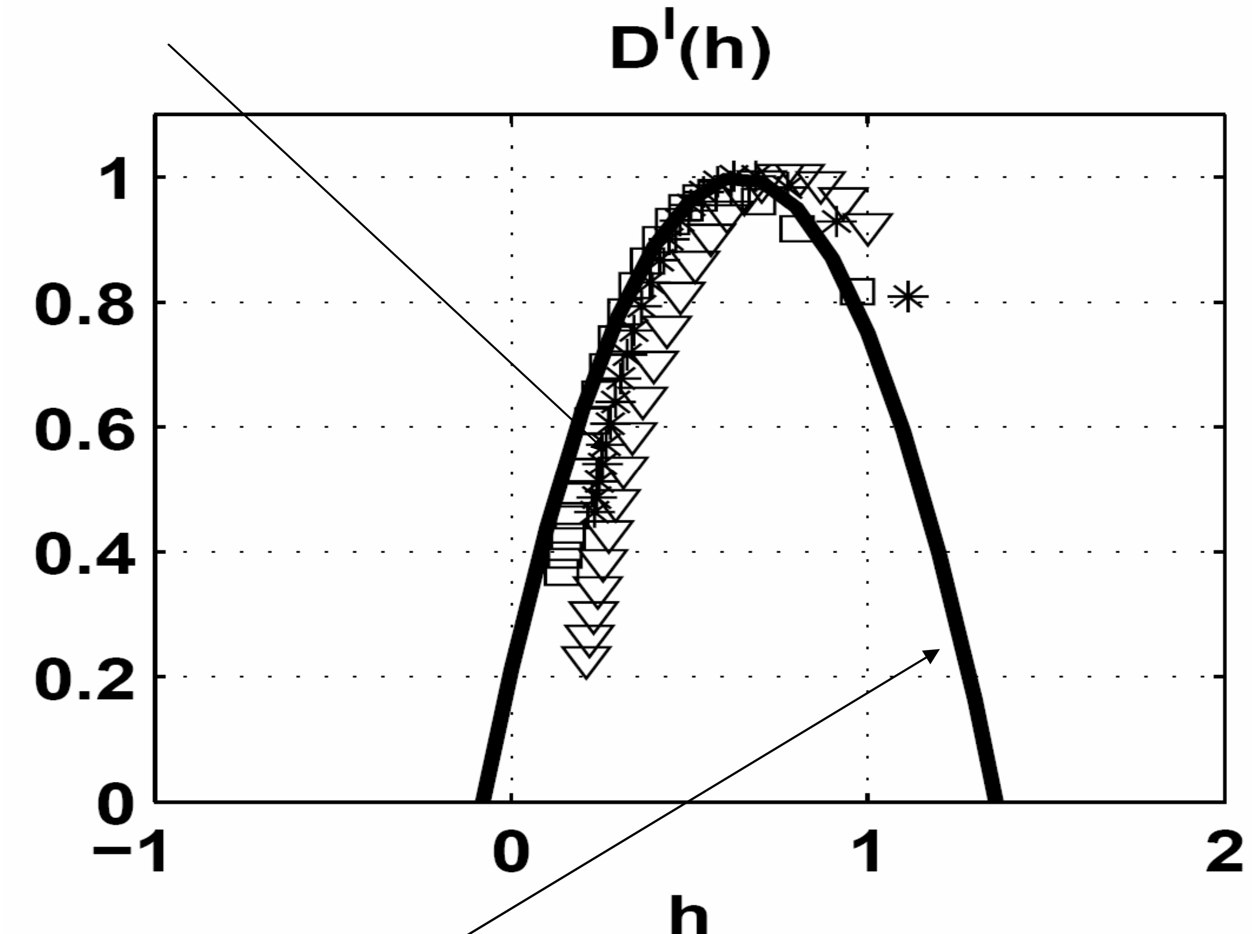
Estimates of c_n : WTMM of Rain Intensity with $g^{(2)}$

	c_0^I	c_1^I	c_2^I	c_3^I
Rain 6	0.98 ± 0.02	0.64 ± 0.03	0.26 ± 0.04	~ 0
Rain 5	0.97 ± 0.02	0.55 ± 0.05	0.38 ± 0.05	~ 0
Rain 4	0.99 ± 0.02	0.62 ± 0.03	0.35 ± 0.15	~ 0
Rain 1	1.00 ± 0.02	0.14 ± 0.03	0.30 ± 0.08	~ 0

$$c_0^I = c_0^c; \quad c_1^I = c_1^c - 1; \quad c_2^I = c_0^c; \quad \ldots$$

Rain 6 Intensity: $D^I(h)$

WTMM Partition Function



Cumulants using WTMM

□ → $g^{(2)}$
* → $g^{(3)}$
∇ → $g^{(4)}$

Two-point Correlation Analysis

$$C(a,\Delta x) = \left\langle \left(\ln|(T_a(x)| - \overline{\ln|(T_a(x)|} \right) \left(\ln|(T_a(x+\Delta x)| - \overline{\ln|(T_a(x+\Delta x)|} \right) \right\rangle$$

It can be shown that if:

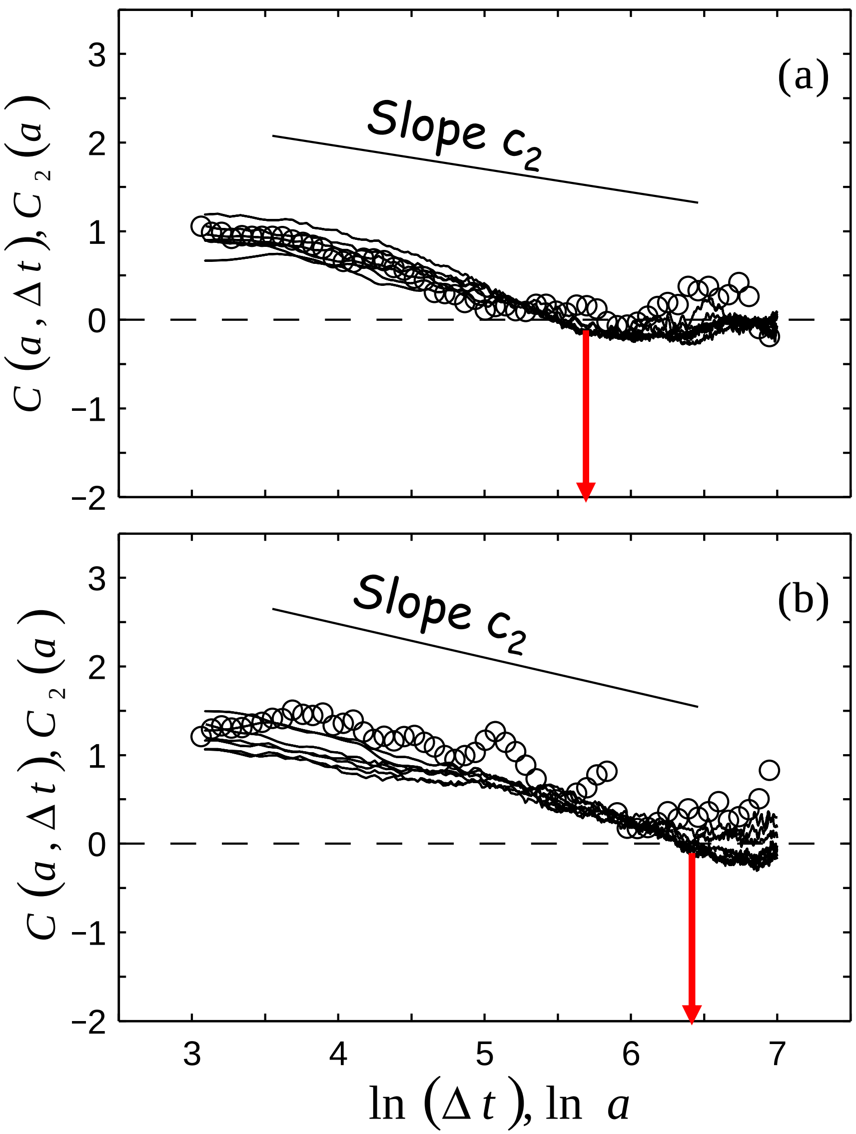
$$C(a,\Delta x) \sim c_2 \ln \Delta x \qquad \Delta x > a$$

➔ multifractal with long-range dependence consistent with that of a multiplicative cascade

CWT with $g^{(2)}$ on rainfall intensity

Rain 6

Rain 5



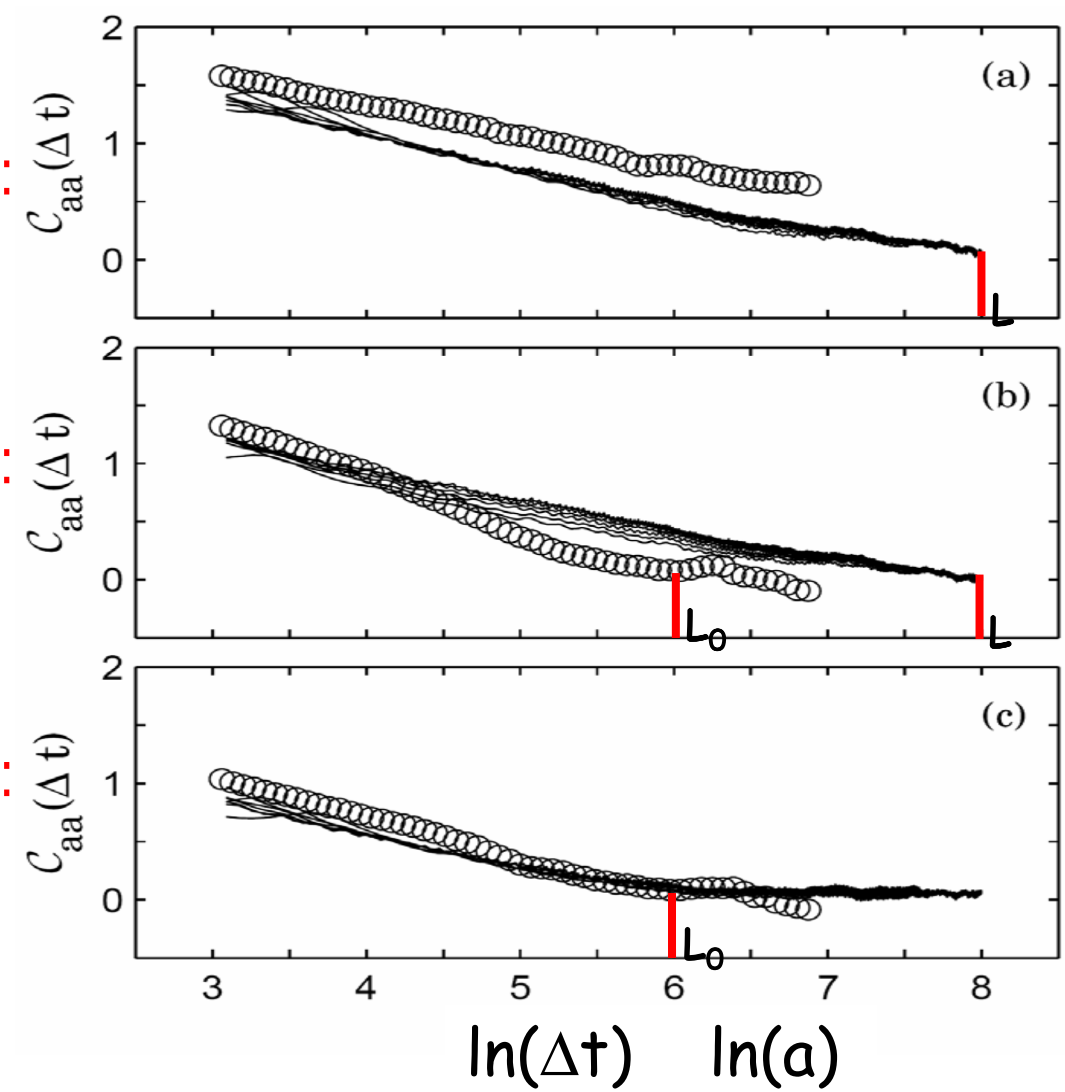
- Long range dependence
- Integral scale $\approx$ 1 -2 hrs
- Consistent with a "local" (within storm bursts) multiplicative cascade

Further evidence of a local cascading mechanism

Case 1 :

Case 2 :

Case 3 :



FIC (L)

LFC (L_0) + FIC (L)

LFC (L_0) + FIC (L_0)

Conclusions

1. Rainfall fluctuations exhibit multifractality and long-range dependence between the scales of ≈ 5 min and ≈1-2 hrs, which coincides with the duration of storm pulses.

2. Storm pulse duration ≈ integral scale in fully developed turbulence; from one "eddy" (storm pulse) to another statistics are not correlated

3. The dynamics within each storm pulse, are consistent with a multiplicative cascade implying a "local cascading mechanism" as a possible driver of the underlying physics.

4. Rainfall fluctuations exhibit a wide range of singularities with $\langle h \rangle = 2/3$ and $h_{min} \approx -0.1$, h_{max} 1.3 $\Rightarrow \exists$ regions where the process is not continuous ($h < 0$) and regions where the process is differentiable once but not twice ($h > 1$)

5. The intermittency coef. $c_2 \approx 0.3$ is much larger than that of turbulent velocity fluctuations ($c_2 \approx 0.025$ longitudinal and $c_2 \approx 0.004$ transverse) and of the same order found in passive scalars, enstrophy ($c_2 \approx 0.3$) and energy dissipation ($c_2 \approx 0.2$) (Frisch, 1995; Kestener+ Arneodo, 2003). The physical interpretation w.r.t rainfall is not clear.

Ref: Scaling behavior of high resolution temporal rainfall: new insights from a wavelet-based cumulant analysis, *Physics Letters A*, 2005

END